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462

FIRST LESSONS

IN

ALGEBRA,

IN WHICH

THE ELEMENTS OF THE SCIENCE ARE FAMILIARLY EXPLAINED.

BY SAMUEL ALSOP,

AUTHOR OF "TREATISE ON ALGEBRA."

NEW EDITION.

PHILADELPHIA:
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OFFICE OF THE CONTROLLERS OF PUBLIC SCHOOLS,
FIRST SCHOOL DISTRICT OF PENNSYLVANIA.
Philadelphia, May 14th, 1851.

AT a meeting of the Controllers of Public Schools, First District of Pennsylvania, held at the Controllers' Chamber, on Tuesday, May 13th, 1851, the following Resolution was adopted:—

Resolved, That "Alsop's First Lessons in Algebra" be introduced as a class-book into the Grammar Schools of this District.

(Signed) ROBERT J. HEMPHILL,
Secretary.

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PREFACE.

In the following elementary treatise on Algebra, the author has endeavored to present the subject in a manner so simple that a child of ordinary intellectual power, who has acquired a knowledge of Arithmetic, can fully comprehend it. He has sought to explain, concisely but clearly, every principle presented, so as to enable such a pupil to *surmount* every difficulty as it occurs; but his aim has not been to make a treatise on Arithmetic, in disguise, nor to *remove* every difficulty from the path of the student, leaving nothing to exercise and strengthen his mental powers. By thus confining himself within the real bounds of the science, and avoiding long dissertations on unimportant subjects, and tedious explanations which a teacher could give much better orally, in a few words, to the very small number of pupils for whom they may be thought necessary, the author has been enabled to comprise within a small compass a larger amount of real algebraic matter than will be found in many treatises of much larger dimensions.

A very large number of examples, illustrative of the principles elucidated, are given. These have been, so far as practicable, made progressive in their character, so that the mind of the pupil shall be gradually led from those which are simple to others presenting greater difficulties, and designed to test the advancement he has made; none, however, being introduced which are deemed inconsistent with the elementary character of this work.

The Binomial Theorem, it will be observed, is introduced early in the work. The manner of illustrating it is, in some respects, novel; and, it is believed, presents the reasons for the rules in a plain and compre-

hensive manner, though it can hardly be considered a demonstration of this most important theorem. Such a demonstration would lead beyond the limits that would be desirable or proper in a treatise designed for beginners. Those who wish to pursue the subject further will find it fully treated of in the author's larger work.

In many modern elementary treatises on Algebra, the student is directed to obtain the roots of a quadratic equation by substituting the coefficients in a formula. This method, which is convenient and useful for one who has become fully versed in the principles of the science, is, in the opinion of the author, very objectionable for a beginner, since by adopting this course he loses the proper idea of the subject, comes to regard the solution as a mere mechanical process, and too frequently forgets entirely the processes of which the formula is the result. Every pupil should, therefore, in the earlier stages of his studies, be required to perform the operation in full. In the following pages, the three rules commonly employed are given separately, accompanied with a number of examples fully sufficient to familiarize the student with their application.

In the chapter on Proportion and Progression are presented all the more important elementary principles of these subjects, illustrated by examples. The articles on Proportion constitute a pretty full treatise on this important subject—sufficiently so to prepare a student for the study of "Legendre's Geometry," or any other treatise in which proportion is considered algebraically. Many of the demonstrations will be found to be new, and very much simplified.

As now published, it is hoped that this little treatise will prove acceptable to teachers, and aid in the advancement of sound mathematical knowledge in our schools.

WILMINGTON, DEL., 8th Month, 1853.

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FIRST LESSONS IN ALGEBRA.

CHAPTER I.

DEFINITIONS.

§ **1.**—1.—Algebra is the science of computing by arbitrary characters. In Arithmetic, our calculations are necessarily restricted by the system of notation there employed. But by the assistance of algebraical symbols, the various results may be generalized, and thus instead of a problem being confined to a particular case, it becomes the enunciation of a general truth, of which the arithmetical truth forms but a single example.

2. Thus if we take any numbers, as 7 and 11, we shall find that the addition of their sum and difference gives 22, which is *twice the greater*. By trying several other numbers, this will always be found to be the case. It might, therefore, be supposed, with some degree of certainty, that the principle is always true. Any difficulty that arises respecting the truth of the problem, is at once removed by assuming two arbitrary numbers, as a and b, and applying to them the rules of addition hereafter to be developed. It will thus be seen that the result produced by adding their sum and difference will, supposing a to be the greater, be *twice* a. Now as a and b may represent any numbers whatever, a being greater than b, the principle is established in all its generality.

3. The quantities employed in algebra, and the operations upon these quantities, being represented by symbols, it will be necessary first to understand the signification of those in most common use.

4 To represent the quantities employed, the letters of the alphabet are generally used, the initial letters, a, b, c, &c. being *most* frequently taken to signify *known* quantities, and the *final* letters, z, y and x, those whose value is to be determined.

It must be kept in mind that these letters are always the representatives of numbers, and not strictly of quantities. Thus, if we say, let x be the length of a line, or the price of an article, it must be understood as representing the number of feet, inches, &c. in the length, or the number of cents or dollars, &c. in the price.

5. With most of the symbols of *operation* the student has become familiar in Arithmetic. Those necessary to understand at the commencement of Algebra are the following.

6. The sign $+$ *plus*, being the symbol of *addition*, is placed between two quantities, when it is intended to represent their sum. The expression $a + b$ is read *a plus b*, and signifies that the quantity represented by b is to be added to that represented by a. So, again, $a + b + c$ signifies the sum of the quantities of which the three symbols, a, b and c are the representatives.

7. The sign $-$ *minus*, is the symbol of subtraction, and indicates that the *latter* of the quantities between which it is placed is to be subtracted from the *former*. Thus, $m - n$ is read *m minus n*, and signifies that the quantity represented by n is to be subtracted from that represented by m.

8. Whenever a quantity is preceded by the sign $-$, it is considered as a *subtractive* quantity. A number or symbol thus distinguished is called a *negative* quantity, those preceded by $+$, and those not preceded by any sign being called *positive* or *affirmative* quantities.

9. The signs $+$ and $-$ are often called the *affirmative* and *negative* signs.

10. The symbol $=$ *equal*, expresses *equality*. Thus, $a = b$ is read *a equals b*; $6 + 4 = 10$, is read 6 *plus* 4 *equals* 10.

11. An expression representing the equality of two quantities or assemblages of quantities, is called an *equation*, the quantities which are considered as equal being the *sides* or *members* of the equation.

12. The sign $\times$ is used to express multiplication. Thus, $a \times b$ signifies a multipled by b. Multiplication is also represented by placing a dot between the factors, or by writing them in connection as in a word. Thus, $a \times b$, $a.b$, and ab, all express the product of a and b. Similarly, $2\,a$ and $3\,x$ represent twice the quantity a and three times x.

13. A number or letter connected with a quantity, to indicate the number of times the latter is repeated, is called a *coefficient*. Thus, $4\,a$ is the coefficient of x in $4\,ax$, or if ax be considered the primary quantity, 4 is the coefficient.

14. Division is expressed by placing the sign $\div$ between the divisor and dividend, the latter being written first, or by placing the divisor beneath the dividend, as in a fraction. Thus, $m \div n$ and $\frac{m}{n}$, both represent the quotient arising from dividing m by n.

15. A vinculum $\overline{\quad}$, or parenthesis ( ), is used to collect several quantities into one. Thus, $7 \times (4 + 5)$ signifies that the sum of 4 and 5 is to be multiplied by 7.

16. Three dots, thus $\therefore$ signify *therefore*.

17. Quantities, the parts of which are not separated by $+$ or $-$, are called *simple* or *monomial* quantities. $5\,a$, $3\,bc$, $\frac{4\,a}{5\,d}$, are all simple quantities.

18. *Compound quantities*, or *polynomials*, consist of two

or more simple quantities connected by the signs $+$ or $-$. Thus, $4a + 3b$, $7ac + 4bc - d$, are compound quantities.

19. The simple quantities which constitute compound quantities are called the *terms*.

20. A compound consisting of two terms is a *binomial;* one containing three terms a *trinomial*, &c.

21. The *power* of a number or root is the product of any number of factors each equal to said root. The number of factors employed gives name to the power. Thus $2 \times 2 = 4$ is the second power (or square) of 2. $2 \times 2 \times 2 = 8$ is the third power (or cube) of 2. So also aaa is the third power of a, or a's cube; $xxxx$ is the fourth power of x, or x's fourth power.

22. The index or exponent of a power is a figure placed over the root to show more concisely the number of factors. Thus, $xxx = x^3$, 3 being the index.

23. The symbol $\sqrt{\ }$, called the *radical sign*, placed before a quantity indicates that some root of it is to be extracted. Thus, $\sqrt{4}$ is the square root of 4. $\sqrt[3]{a}$ is the cube root of a. $\sqrt[4]{x}$ is the fourth root of x.

24. Roots are likewise expressed by fractional indices: thus, $a^{\frac{1}{2}}$ is the square root of a, $x^{\frac{2}{3}}$ is the cube root of x's square, the denominator of the index expressing the root and the numerator the power.

25. A root which can be accurately expressed in numbers, either whole or fractional, is a *rational quantity*. Thus, $\sqrt{4}$, $\sqrt[3]{8}$, $\sqrt{\frac{9}{16}}$ are rational quantities.

26. An *irrational quantity* or *surd* is a root which cannot be accurately expressed in numbers. Thus, $\sqrt{3}$, $\sqrt[3]{4}$ are irrational quantities.

27. The *reciprocal* of a number is the quotient produced by dividing unity by it. Thus, $\frac{1}{a}$ is the reciprocal of a, $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$, &c.

28. *Similar quantities* are those which consist of the same letters or combinations of letters. Thus, $4\,ac$, $5\,ac$, $7\,ac$, are similar quantities.

29. *Dissimilar quantities* consist of different letters or combinations of letters. Thus, $4\,a$ and $5\,x$, or $7\,aac$, $5\,ac$ and $6\,acc$, are dissimilar quantities.

30. The *common multiple* of any numbers, or quantities, is a number or quantity into which they will all divide without remainder. Thus 24 is a common multiple of 2, 3, 4, and 6.

31. The *common measure* of any numbers or quantities, is the number or quantity which will divide them all without remainder. Thus, 7 is a common measure of 14, 28, and 35.

32. Numbers or quantities which have no common measure are *prime* to each other.

Examples of the use of the signs.

§ 2. If $a = 9$, $b = 5$, $c = 3$, $d = 1$, and $e = 0$, what are the values of the following expressions?

1. $4\,ab + c$
   $4\,ab + c = 4 \times 9 \times 5 + 3 = 180 + 3 = 183.$
2. $7\,ac + 5\,bc - 2\,d$
   $7\,ac + 5\,bc - 2d = 7 \times 9 \times 3 + 5 \times 5 \times 3 - 2 \times 1 = 189 + 75 - 2 = 264 - 2 = 262.$
3. $5\,bc + 2\,d.$
4. $6\,ab - 3\,cd + e.$

5. $3\,abc + 4\,bd - 7\,ae$.

6. $5\,bcd - ab + 6d$.

7. $\dfrac{4\,b - cd.}{3\,a.}$

8. $\dfrac{5\,bc - 4\,ae + 6\,c}{2\,bc + 3\,ad.}$

9. $\dfrac{7\,cd + 4\,ab - 3\,ac}{7\,ab - 2\,c.}$

10. $4\,a\,(2\,b + 3\,cd) - 3\,(a + b)$.

In this example the value of $2\,b + 3\,cd$ is to be multiplied by $4\,a$, and 3 times $a + b$ taken from the product. The operation, therefore, stands thus:—

$$4\,a\,(2\,b + 3\,cd) - 3\,(a + b) = 36 \times (10 + 9) - 3 \times (9 + 5) = 36 \times 19 - 3 \times 14 = 684 - 42 = 642.$$

11. $9\,c\,(3\,ac + d) - 2\,c\,(4\,ad - 2\,bc)$.

12. $\dfrac{6\,a + 5\,b\,(c + d)}{7\,ca - 4\,b.}$

13. $\dfrac{14\,b - 2\,ad + 6\,cd}{9\,(3\,ac - 2\,bd).}$

14. $(3\,a + 2\,b)\,(5\,c - 3\,d) \div (3\,a + c)\,b$.

CHAPTER II.

ADDITION.

§ **3.** By the addition of algebraic quantities, is meant collecting them together; performing with each the operation indicated by its sign. This operation can of course only be performed with quantities which are of similar natures.

Those which are not so, must be connected by their signs. It is evident that 5 yds. + 3 yds. make 8 yds., but the sum of 5 lbs. and 3 yds. can be expressed only by connecting them with the sign +.

§ 4. When the quantities are all positive, and of the same kind, no difficulty in summing them can occur, since the operation is manifestly performed as in Arithmetic. Thus,

$$6 + 4 + 3 = 13, \text{ and } 6x + 4x + 3x = 13x$$
as much as 6 yds. + 4 yds. + 3 yds. = 13 yds.

§ 5. Dissimilar quantities, as above remarked, can only be added by connecting them by their signs. Thus, if Thomas receives from one man \$5, from another 3 yds., from a third 2 yds., and from a fourth \$12, he receives altogether, \$17 and 5 yds., which may be written \$17 + 5 yds.

So also, $5a + 3x + 2x + 12a = 17a + 5x$.

§ 6. If any of the quantities are negative, they must be subtracted from the sum of the positive quantities. Thus, let the value of the expression

$$75 - 37 - 24$$

be required. This expression evidently means 75 diminished by 37, and the result diminished by 24. We therefore have

$$75 - 37 - 24 = 38 - 24 = 14.$$

Similarly, if the value of

$$47 - 14 + 17 + 23 - 16$$

be required, we shall have

$$47 - 14 + 17 + 23 - 16 = 33 + 17 + 23 - 16$$
$$= 50 + 23 - 16 = 73 - 16 = 57.$$

Required, the value of $76 - 48 + 10 - 92$.

$$76 - 48 + 10 - 92 = 28 + 10 - 92 = 38 - 92.$$

This last result requires the number 92 to be taken from 38, an operation manifestly impossible. The greatest number which can be taken from 38 being itself, it is evident

that the above expression requires us to deduct 54 more than can be done. This result is written -54: the sign $-$ implying an unperformed subtraction. In all cases, this sign when standing before an isolated quantity has this signification. Sometimes it also indicates an impossibility in the conditions of the problem as stated, and sometimes a mere change of direction. Thus, suppose a person set out without any money to collect some debts and likewise to pay some: he obtained from A \$100, from B \$50; he then paid C \$75 and D \$90: how much had he left? The formula would stand thus:

$$100+50-75-90=150-75-90=75-90=-15.$$

The -15 indicates that the last payment is impossible.

Again, suppose A borrows \$10 of B, then pays \$7 to C, borrows \$15 from D, and finally pays \$20 to E, how much have his funds been increased?

$$10-7+15-20=3+15-20=18-20=-2.$$

Here the -2 does not indicate any impossibility in the problem, since it is assumed that A had some money when he began. It merely shows that the sum has not been *increased*, but *diminished* by \$2.

As another example, if John started from Philadelphia towards Pittsburg, and travelled on the first day 40 miles, on the second returned 27 miles, on the third he again proceeded 30 miles, and returned on the fourth 50 miles, how far on his journey had he advanced?

$$40-27+30-50=13+30-50=43-50=-7.$$

The -7 indicates that he is 7 miles farther from Pittsburg than when he started.

§ 7. From the above examples we find the following

Rule for addition of like quantities.

Collect the positive quantities into one sum, and the negative into another, and take the difference of the two results, prefixing the sign of the greater.

EXAMPLES.

Ex. 1. Add $14+7-3+16-8=37-11=26$ *Ans.*

Ex. 2. Collect $6a-4a+8a-7a=14a-11a=3a$ *Ans.*

Ex. 3. Collect $5ax-7ax-9ax+10ax=15ax-16ax=-ax$.

Ex. 4. Collect $16ab-4ab+3ab-7ab$.

Ex. 5. Collect $14bx-20bx-10bx+25bx$.

Ex. 6. Collect $37xyz+10xyz-15xyz+4xyz$.

Ex. 7. Collect $15abx-14abx+17abx-12abx$.

Ex. 8. Collect $19bcd-25bcd+8bcd-7bcd$.

Ex. 9. Collect $5xyz-17xyz+12xyz$.

Ex. 10. Collect $15axy-23axy+12axy-4axy$.

§ 8. In what precedes, the quantities of which the amount was required, were all similar. As was remarked, (§5), it is only of such quantities that the sum can really be obtained. If unlike quantities be given, the only method is to

Collect those which are alike into several amounts, as in last article, and join them by their proper signs.

EXAMPLES.

Ex. 1. Let the sums of the following quantities be required, viz. $6a, +4x, -3a, +5x, -3x, +7a$.

Here the a's are $6a-3a+7a=10a$
and the x's are $4x+5x-3x=6x$
the result is therefore $10a+6x$.

Ex. 2. Collect $7ax-4ab+3ab-5ax-6ax+7ab$.

First, $7ax-5ax-6ax=-4ax$
Second, $-4ab+3ab+7ab=6ab$
the sum required is therefore $6ab-4ax$.

Ex. 3. Collect $5xy+7a+4b-6xy+3a-5b$
Ans. $10a-xy-b$

Ex. 4. Collect $7ax-4bx+10ax-6ax+4bx$.
Ans. $11ax$.

Ex. 5. Collect $6xy-12xz+4xy-3xy+7xz-4xz$.

Ex. 6. Collect $5xyz - 6xy + 8xyz - 7xy + 11xy$.

Ex. 7. Collect $15ax + 12ab - 7ax - 9ab - 10ab$.

Ex. 8. Collect $19bc - 25bc - 7ax - 4bc + 11ax$.

Ex. 9. Add
$$\begin{array}{r}
6ax - 4ab - 5ac \\
7xy + 4ax - 5ab \\
6ax + 7ac - 5xy \\
3xy + 17ab - 8ac \\
\hline
16ax + 5xy + 8ab - 6ac
\end{array}$$

Ex. 10. Add
$$\begin{array}{r}
6a - 7b + 8x \\
4x - 6a + 8b \\
9b - 6x + 4a \\
12a - x + b \\
a - 4b + x \\
\hline
\end{array}$$

Ex. 11. Add
$$\begin{array}{r}
12ax + 4xy - 5az \\
17az - 17xy + 15ax \\
10az - 11ax + 4xy \\
3ax + 9xy - 7az \\
\hline
\end{array}$$

Ex. 12. Add
$$\begin{array}{r}
13yz - 17xy + 4b \\
7b + 11xy - 13yz \\
6xy - 4b + 7xy \\
19yz - 17xy - 11b \\
\hline
\end{array}$$

Ex. 13. Add
$$\begin{array}{r}
3abx - \frac{1}{2}bx + 5ax \\
7\frac{1}{2}ax - 3\frac{1}{4}abx + 15bx \\
-6bx + 4abx - \frac{1}{2}ax \\
5ax - 6abx - 4bx \\
\hline
\end{array}$$

Ex. 14. Add
$$\begin{array}{r}
7xy + 3ax - 5b \\
4b - 17xy + 4ax \\
-6ax + 10b - 10xy \\
3xy + ax - b \\
\hline
\end{array}$$

Ex. 15. Add $14ax - 5bc + 4d$, $12bc - 14d + 7bc$, $8ax + 6bc - d$, and $7bc + 12ax - 2d$.

Ex. 16. Add $4x - 16b + 40$, $37 - 12b + 7x$, $12b - 45 - 6x$, and $15x - 23 + 17b$.

Ex. 17. Add $6ab - 6xz + 14xyz + 27$,
$35xz - 17xyz + 8$, $16ab + 25$,
and $15xyz - 42 + 15ab + 10xz$.

SUBTRACTION.

§ 9. Subtraction being the reverse of addition, it is evident that the terms of a polynomial which is *subtractive* must be applied in an opposite manner from what they would be were it *additive*. Thus, since $4a + 5x - 3b$ added to $7a - 4x + 8b$ makes $11a + x + 5b$, it follows, that $4a + 5x - 3b$ subtracted from $11a + x + 5b$ must leave for remainder $7a - 4x + 8b$.

Now from the principles of addition it is evident that if

$$\begin{array}{ll} \text{to} & 11a + x + 5b \\ \text{we add} & -4a - 5x + 3b \\ \hline \text{we obtain} & 7a - 4x + 8b. \end{array}$$

Hence, to subtract $4a + 5x - 3b$, we change the signs, and then apply the rules of addition.

Rule.

Change the signs of the quantities to be subtracted, and then apply the rules of addition.

Examples.

Ex. 1. From $7a - 4b$ take $3a - 2b$

$$\begin{array}{r} 7a - 4b \\ 3a - 2b \\ \hline 4a - 2b \end{array}$$

Ex. 2. From $9a + 4x - 7b$ take $4a - 3x + 2b$

$$\begin{array}{r} 9a + 4x - 7b \\ 4a - 3x + 2b \\ \hline 5a + 7x - 9b \end{array}$$

Ex. 3.

$$\begin{array}{ll} \text{From} & 12ax - 4ab + 3bc \\ \text{take} & 10ax - 3bc - 4ab \\ \hline & 2ax + 6bc \end{array}$$

Ex. 4.

$$\begin{array}{ll} \text{From} & 17bc + 6ac - 4x \\ \text{take} & 5bc + 3ac - 2x \\ \hline \end{array}$$

Ex. 5. From $9a - 6bc + 4x$ take $7a - 4bc - 3x$.

Ex. 6. From $11x - 4ay + 6bc$ take $5x - 4bc + 3ay$

Ex. 7.

$$\begin{array}{ll} \text{From} & 12yz - 14xx + 16d \\ \text{take} & 7xx - 4yz + 3d \end{array}$$

Ex. 8. From $19\,xy - 7\,yz + 14\,b$
take $16\,b - 3xy + 7\,yz$.

Ex. 9. From $12\,abc - 15\,bd + 4\,e$
take $5\,e - 11\,bd + 10\,abc$.

Ex. 10. From $6\,ax - 4\,bc - 7\,d + 8\,e$, take $4\,ax - 4\,e + 3\,bc - 5\,d$.

Ex. 11. From $9\,abc - 4\,ac - 7\,bcd + 9\,e$, take $4\,ac + 9\,bac + 7\,bcd + 2e$.

Ex. 12. From $4\,bx - 5\,xy + 7\,ad + 3\,be$, subtract $8\,be - 3\,bx - 4\,xy - 2\,ad$.

Ex. 13. From $15\,aby - 7\,acx + 5\,bcd + 8\,ay$, take $14\,ay - 3\,aby + 8\,acx - 15\,bcd$.

Ex. 14. From $5x^2 - 3\,ay + 4\,bcx$, take $17\,bcx - 4\,ay - 3x^2$.

Ex. 15. From the sum of $6\,x - 3\,b + 4\,a$, $7\,a - 5\,b - 6\,x$, and $5\,x - 12\,b - 4\,a$, take the sum of $7\,x - 3\,a + 11\,b$, $4\,x - 16\,a + 8\,b$, and $3\,b - 11\,a + 14\,x$.

Ex. 16. From the sum of $9\,x - 3\,a + 15$, $60 + 8\,a - 5\,x$, and $7\,a - 16x + 10$, take $19\,a - 7\,x + 5$.

Ex. 17. From the sum of $5\,ax - 7\,bc + 4\,dx - y$, $6\,ax - 4\,y + 8\,bc - 2\,dx$, and $3\,bc - 7\,ax - 5\,y + 8\,dx$, take the sum of $9\,dx - 5\,ax + 8\,bc - 7\,y$, and $11\,y + 15\,bc - 4\,ax - 5\,dx$.

Ex. 18. Add $5\,ay + 6\,xy - 3\tfrac{1}{2}\,bc$, $4\tfrac{1}{2}\,bc - 7\,ay + 11\,xy$, $5\,bc - 4\,ay - 7\,xy$, and from the result subtract the sum of $8\,xy - 14\,bc + 27\,ay$, $8\,bc - 14\,ay + 3\,xy$, and $2\,bc - 13\,xy - 7\,ay$.

§ **10.** When any compound quantity is to be subtracted, it is evident from what precedes, that the signs of such quantity must be changed before the terms are written down separately. Thus $4\,a - (3\,b - c)$ indicates that the whole quantity $3\,b - c$ is to be subtracted from $4\,a$; if therefore, it be desired to avoid the parenthesis, it will be necessary to change the signs both of the $3\,b$ and c. The expression will therefore stand thus, $4\,a - 3\,b + c$, and not $4\,a - 3\,b - c$. So also $6\,a - 4\,b - (4\,g - 2\,x + d) = 6\,a - 4\,b - 4g + 2\,x - d$.

EXAMPLES.

Express without parenthesis the following:

1. $7a - (b - c)$.
2. $6ax - (4b^2 + 5cx)$.
3. $5by - (4xy + 8ab - 2c)$.
4. $7ax - 4ab - (18ac - 4ax + b)$.
5. $3a^3 - (5ay + 4bcy - 7ax - d)$.

MULTIPLICATION.

§ 11. It has already been said (Def. 12, p. 9) that the product of two quantities, such as a and b, is expressed by $a \times b$, $a \,.\, b$, or more simply by ab.

In the same manner the product of any number of factors is expressed. Thus, $a \times b \times c \times d$ is written $abcd$.

So likewise, $5ab \times 3cd = 5ab\,3cd$. But since it is indifferent what order is maintained amongst the factors, the result may be written

$$5 \times 3 \times abcd, \text{ or } 15\,abcd.$$

Hence, *To multiply monomials, multiply the numerical parts,* or coefficients, *and to the product annex the product of the literal parts.*

EXAMPLES.

Multiply $4ac$ by $3bd$.	*Ans.*	$12\,abcd$.
" $3ad$ by $5ac$.	"	$15\,aacd$.
" $4aax$ by $7ay$.	"	$28\,aaaxy$.
What is the value of $7ay \times 12bx \times 6ab$.	"	$504\,aabbxy$

Reduce the following, viz.:

$$6ax \times 3ay \times 4bc =$$
$$8by \times 6abx \times 2cd =$$
$$5aaabbbx \times 7aaabxx =$$
$$7abxxy \times 3abxx \times 4x =$$
$$12aabc \times 3abccc =$$
$$16aad \times 2bbc \times 7abed =$$
$$13aabcc \times 6abcx =$$
$$5aac \times 4aabbc \times 7aabx =$$

§ **12.** In the above examples such expressions as aaa, bb, &c. frequently occur.

Now we have learned in arithmetic that the
product of two equal factors is the square of one of them,
" three " " cube "
" four " " fourth power, &c.

Consequently aa is the square or 2d power of a.
aaa is the cube or 3d power of a, &c.

In order to render the expressions more concise, the number of factors is indicated by putting a small figure over the root, and a little to the right; thus, a^2 is written for aa, and is read a's square, or the square of a.

Similarly, $a^3 = aaa$, and is read a's cube, or the cube of a, so a^4, a^6, a^7, are respectively the same as $aaaa$, $aaaaaa$, and $aaaaaaa$, and are read a's fourth, a's sixth, and a's seventh power. See Def. 21, page 10.

§ **13.** The figure which thus indicates the power, is called the *exponent*, or *index*, and represents the number of equal factors that are multiplied together. Thus, when we say

$$4^3 = 64, \text{ we mean } 4 \times 4 \times 4 = 64.$$

The indices must be carefully distinguished from the *coefficients*, since these express only successive additions, while the former represent successive multiplications. Thus,

$$3a = a + a + a, \text{ while } a^3 = a \times a \times a.$$

§ **14.** From what has been said above, it is easy to write the results in the 11th section more concisely. The second, third, and fourth may be written thus, $15\,a^2cd$, $28\,a^3xy$ and $504\,a^3b^2xy$.

The student will thus simplify the remaining results in that article.

§ **15.** Since $x^4 = xxxx$, and $x^5 = xxxxx$
it is evident that
$$x^4 \times x^5 = xxxx \times xxxxx$$
$$= xxxxxxxxx = x^9.$$

Similarly we should find that

$$x^2 \times x^3 = x^5,\ x^7 \times x^5 = x^{12}, \text{ \&c.}$$

Hence, *To multiply different powers of the same root, we add their indices.*

Examples.

Ex. 1. $7a^2 \times 5a^3 = 35a^5$.

Ex. 2. $5a^2x^2 \times 4a^3x^3 = 20a^5x^5$.

Ex. 3. $3a^2x^4 \times 6a^3x = 18a^5x^5$.

Ex. 4. $9a^4x^2 \times 7a^3x =$

Ex. 5. $7ab^3c^2 \times 4a^3bc^3 =$

Ex. 6. $12a^4bc^4 \times 3a^5bc^5 =$

Ex. 7. $3a^2x^3b \times 5ax^4by^4 =$

Ex. 8. $15a^3bx^4 \times 3a^2x^9b^7 =$

Ex. 9. $12a^5b^4c^3 \times 4a^{10}b^2c^7 =$

Ex. 10. $17a^2b^3c^4 \times ac^3 =$

Ex. 11. $8x^2y^3z \times 9xy^4z^3 =$

Ex. 12. $15a^3x^2y^5 \times 3b^7x^6y^8 =$

Ex. 13. $9b^2d^3y^2 \times 4a^3d^2 =$

Ex. 14. $15b^3d^2e^4 \times 5b^3d^9e^{10} =$

MULTIPLICATION OF POLYNOMIALS.

§ **16.** If $a + b$ is to be multiplied by any number as 3, it is equivalent to adding three quantities, each equal to $a + b$; the result will evidently be $3a + 3b$, which consists of three times the first quantity plus three times the second. Had the expression been $a - b$, the result would have been $3a - 3b$. This would be equally true if the multiplicand consisted of more than two terms, or if the multiplier were any other number. Hence,

To multiply a polynomial by a positive multiplier, we multiply each term separately, and connect the results by the signs with which the several terms were affected in the multiplicand.

Examples.

Ex. 1. $(5ax - 6a^3b - 3ac) \times 4a = 20a^2x - 24a^4b - 12a^2c$.

Ex. 2. $(7xy^2 - 4az + 3b) \times 6a^2x^2 = 42a^2x^3y^2 - 24a^3x^2z + 18a^2bx^2$.

Ex. 3. $(4ab^2 - 5a^2c + b^2) \times 7a^2b^3c =$

Ex. 4. $(8a^2x - 15ax^2 + 15) \times 12a^3x^3 =$

Ex. 5. $(9bc^3 - 8b^3c + b^2c^2) \times 15b^2c =$

Ex. 6. $(12ab^3 - 4d^2x - 5a^2c) \times 4ab^2cx =$

Ex. 7. $(3a^2b^2 - \frac{1}{2}b^3c + 8c^2) \times \frac{1}{2}bc^4 =$

Ex. 8. $(9a^3d - 4ad^2 - 3d) \times 7a^4b =$

Ex. 9. $(\frac{3}{2}x^3y - \frac{4}{3}b^2z + \frac{3}{4}xy^3) \times 12b^3x^2y^2 =$

In the above cases we perceive that a negative quantity multiplied by a positive, gives a negative product.

§ **17.** We shall now proceed to the case in which the multiplier is a polynomial as well as the multiplicand.

Let it be required to multiply $x + y$ by $a + b$. This is evidently requiring us to increase $x + y$, $a + b$ times, which is equivalent to multiplying it by a and also by b, and adding the results. The operation may be arranged thus:

$$\begin{array}{rllll}
 & x & + y & & \\
 & a & + b & & \\
\hline
 & ax & + ay & & \text{product by } a \\
 & & + bx & + by & \text{"} \quad b \\
\hline
ax + & bx & + ay & + by & \text{"} \quad a + b.
\end{array}$$

Similarly if the product of $(x - y)$ by $(a + b)$ were required, the operation would evidently be

$$\begin{array}{llll}
x - y & & & \\
a + b & & & \\
\hline
ax - ay & & & \text{product by } a \\
 & bx - by & & \text{"} \quad b \\
\hline
ax - ay & + bx - by & & \quad (a + b).
\end{array}$$

§ **18.** Had the multiplier been $a - b$, it is evident the first line $ax - ay$, which is a times the multiplicand, would have been too great; and would require to be diminished by b times $(x - y)$ or $bx - by$, which is the second line. But since in subtraction, we change the signs of the subtracting terms, the operation by addition might still be preserved, by writing the terms in the last-mentioned line, with the opposite signs, as below.

$$\begin{array}{llll}
x - y & & & \\
a - b & & & \\
\hline
ax - ay & & & \text{product by } \quad a \\
 & - bx + by & & \text{"} \quad -b \\
\hline
ax - ay & - bx + by & & \text{"} \quad (a - b).
\end{array}$$

By examining the various terms in this operation, we perceive that

$$\begin{aligned} a &\times x = + ax \\ a &\times -y = - ay \\ -b &\times x = - bx \\ -b &\times -y = + by \end{aligned}$$

Hence we derive the following

Rule for Signs.

Like signs in multiplication produce *plus*; *unlike signs* in multiplication produce *minus*.

§ **19.** From the investigation in the last article we obtain the following

RULE FOR MULTIPLYING BY A POLYNOMIAL.

Multiply the multiplicand by the several terms in the multiplier, and add the results.

EXAMPLES.

Ex. 1.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \qquad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

Ex. 2.

$$\begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \qquad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

Ex. 3.

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \qquad - ab - b^2 \\ \hline a^2 \qquad\quad - b^2 \end{array}$$

Ex. 4.

$$\begin{array}{l} a^2 - 2ab + b^2 \\ a + b \\ \hline a^3 - 2a^2b + ab^2 \\ \qquad\quad a^2b - 2ab^2 + b^3 \\ \hline a^3 - a^2b - ab^2 + b^3 \end{array}$$

Ex. 5.

$$\begin{array}{l} 3a^3b - 2ab^3 + b^4 \\ 2ab + b^2 \\ \hline 6a^4b^2 - 4a^2b^4 + 2ab^5 \\ \qquad + 3a^3b^3 - 2ab^5 + b^6 \\ \hline 6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6 \end{array}$$

Ex. 6. Multiply $x + y$ by $x + y$. *Ans.* $x^2 + 2xy + y^2$.

Ex. 7. Multiply $x + y$ by $x - y$ *Ans.* $x^2 - y^2$.

Ex. 8. Multiply x^2+y^2 by x^2+y^2. *Ans.* $x^4+2x^2y^2+y^4$.

Ex. 9. Multiply $x^2 + y^2$ by $x^2 - y^2$. *Ans.* $x^4 - y^4$.

Ex. 10. Multiply $x^2 - 2xy + y^2$ by $x - y$.
Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

Ex. 11. Multiply $x^3+3x^2y+3xy^2+y^3$ by $x^2+2xy+y^2$.
Ans. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

Ex. 12. Multiply $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.
Ans. $x^4 - 2x^2y^2 + y^4$.

Ex. 13. Multiply $a^4 - a^3y + a^2y^2 - ay^3 + y^4$ by $a + y$.
Ans. $a^5 + y^5$.

Ex. 14. Multiply $5x^2-3ax+4a^2$ by $7x^2+3ax-5a^2$.
Ans. $35x^4 - 6ax^3 - 6a^2x^2 + 27a^3x - 20a^4$.

Ex. 15. Multiply $6a^3 - 7a^2x + 4ax^2 - 4x^3$
by $5a^2-4ax-3x^2$.
Ans. $30a^5-59a^4x+30a^3x^2-15a^2x^3+4ax^4+12x^5$.

Ex. 16. Multiply $9a^4-5a^3b+7a^2b^2-6ab^3+7b^4$
by $3a^2-2ab+b^2$.
Ans. $27a^6 -33a^5b+40a^4b^2- 37a^3b^3 + 40a^2b^4$
$-20ab^5+7b^6$.

Ex. 17. Multiply $6x^4- 7abx+4bx^2$ by $5x^4-3ax+x^3$.
Ans. $30x^8+ 6x^7+ 20bx^6- 35abx^5-18ax^5$
$+ 4bx^5 - 7abx^4 - 12abx^3+21a^2bx^2$.

Ex. 18. What is the product of $x - 3y$, $x - y$, $x + y$, and $x + 3y$. *Ans.* $x^4 - 10x^2y^2 + 9y^4$.

Ex. 19. What is the product of $a^4+ a^3b+a^2b^2+ ab^3+ b^4$, and $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.
Ans. $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$.

Ex. 20. Multiply $3x^2 - 2xy + 5$ by $4x^2 + xy - 7$.
Ans. $12x^4-5x^3y - x^2- 2x^2y^2+ 19xy - 35$.

DIVISION.

§ **20.** The division of simple quantities can present but little difficulty, since its operations must be the reverse of those of multiplication.

Thus the product of two powers of the same root is found by adding their indices. For example, $a^7 \times a^3 = a^{10}$. Hence, $a^{10} \div a^7 = a^3$, and as the operation will be similar whatever the indices may be, it follows that

To divide different powers of the same root, subtract the index of the divisor from that of the dividend, the remainder is the index of the quotient.

We shall also find that the same rule for signs holds as in multiplication.

Hence, in *Multiplication* and *Division*
Like signs produce *plus*,
Unlike signs produce *minus*.

§ **21.** It is often convenient for beginners to write the divisor beneath the dividend as in a fraction, and cancel the like factors, as in Arithmetic.

Thus the division of

$$27\,a^4b^3c^2 \text{ by } -9\,a^2b^3c$$

may be performed thus,

$$\frac{27\,a^4b^3c^2}{-9\,a^2b^3c} = -3\,a^2c;$$

the common factors 9, a^2, b^3, and c, having been cancelled.

This mode of operation can hardly be recommended, however, except for those persons who have not acquired any facility in calculation, as we may obtain the result in all cases, at least where the quotient is not fractional, by a more simple process. We should divide the coefficients, and then the literal parts, setting them down in order: first, however, having been careful to notice and write the sign with which the quotient will be affected.

In the above example the operation would be as follows:

$$-9\,a^2b^3c\,\underline{)\,27\,a^4b^3c^2}$$
$$-3\,a^2c$$

Thus, unlike signs produce minus; 9 into 27 gives 3, a^2 into a^4 goes a^2, b^3 into b^3 gives 1, and c into c^2 goes c times. The result is therefore as above; the factor 1 not appearing, as it does not affect the result.

EXAMPLES.

Ex. 1. Divide $48x^2$ by $12x$. *Ans.* $4x$.

Ex. 2. Divide $36xy^2$ by $6xy$. *Ans.* $6y$.

Ex. 3. Divide $15a^2xy^3$ by $-3ay$. *Ans.* $-5axy^2$.

Ex. 4. Divide $-27a^2x^4y$ by $9a^2x^2$. *Ans.* $-3x^2y$.

Ex. 5. Divide $-84x^3y^2z$ by $-12xyz$. *Ans.* $7x^2y$.

Ex. 6. Divide $45x^7y^8$ by $9x^4y$. *Ans.* $5x^3y^7$.

Ex. 7. Divide $96a^7b^6c^4$ by $12a^4b^6c^2$. *Ans.*

Ex. 8. Divide $-16a^9x^{13}b$ by $4x^3a^2b$.

Ex. 9. Divide $54b^3c^7d^8$ by $-9bd^7c$.

Ex. 10. Divide $38b^4ca^4$ by $-2b^2ac$.

Ex. 11. Divide $16x^4b^7c^4$ into $64x^7b^9c^{11}$.

Ex. 12. Divide $-49ax^2b^9$ into $98a^4x^{11}b^9$.

Ex. 13. Divide $-2b^2d^2f^6$ into $-18b^4d^2f^7$.

Ex. 14. Divide $-4a^2b^6c^8$ into $-60a^4b^7c^9$.

Ex. 15. Divide $200a^8b^8x$ by $50a^7b^8x$.

Ex. 16. Divide $-17a^2b^9c^7$ by a^2b^7c.

Ex. 17. Divide $14b^{12}fg$ by $2b^{12}f$.

§ **22.** The division of monomials can only be effected when the coefficient of the dividend is a multiple of that of the divisor, and all the letters contained in the divisor are likewise found in the dividend, with indices at least as great as they are affected with in the divisor. In all other cases the quotient can only be expressed fractionally.

Thus if it were required to divide

$$-15a^4x^3y \text{ by } 10a^2x^4y^2,$$

the quotient would be $\dfrac{-15a^4x^3y}{10a^2x^4y^2}$.

which by cancelling the common factors 5, a^2, x^2, and y, is reduced to

$$-\frac{3\,a^2}{2\,xy}.$$

From this example we derive the following

Rule.

Set the divisor under the dividend, and divide the numerator and denominator of the fraction thus formed, by the greatest common divisor of the coefficients, and by such letters as are contained in both terms; or,

Divide both terms by their greatest common measure.

Examples.

Ex. 1. Divide $15\,a^3x^2$ by $-3\,a^2x^3$. *Ans.* $-\frac{5\,a}{x}$.

Ex. 2. Divide $17\,b^2c$ by $-5\,bc^3$. *Ans.* $-\frac{17\,b}{5\,c^2}$.

Ex. 3. Divide $-21\,a^3bx^2$ by $-15\,ab^2x^3$. *Ans.* $\frac{7\,a^2}{5\,bx}$.

Ex. 4. Divide $14\,m^2n^3$ by $7\,mn^4$. *Ans.* $\frac{2\,m}{n}$.

Ex. 5. $-45\,abx^4$ by $21\,a^2bx$. *Ans.* $-\frac{15\,x^3}{7\,a}$.

Ex. 6. Divide $72\,a^3bc^2$ by $12\,a^3bc^2x^2y$.

Ex. 7. Divide $-49\,x^2yz^4$ by $-21\,x^3y^2z$.

Ex. 8. Divide $96\,a^3x^4y^7$ by $12\,a^5x^4y^9$.

Ex. 9. Divide $87\,a^5m^2n$ by $27\,a^4mn^2$.

Ex. 10. Divide $128\,a^3x^4bc^2$ by $28\,ax^5bc^7$.

DIVISION OF POLYNOMIALS.

§ **23.** The principles of long division in arithmetic, apply without any essential change here; care being taken to perform the various multiplications and subtractions according to the rules given in the preceding pages.

Rule.

Arrange the terms of the divisor and dividend according

to the powers of some letter, either commencing with the highest power and regularly descending, or with the lowest and regularly ascending.

Divide the first term of the divisor into the first term of the dividend, for the first term of the quotient. Multiply the divisor by the term thus determined, and subtract the product from the dividend, arranging the terms as above directed.

Divide the first term of the remainder by the first term of the divisor, and so proceed until the operation is accomplished.

Examples.

Ex. 1. Divide $a^2 - 2ax + x^2$ by $a - x$.

$$\begin{array}{l} a - x\,)\,a^2 - 2ax + x^2\,(\,a - x \\ \qquad\quad\; \underline{a^2 - ax\quad} \\ \qquad\qquad\; - ax + x^2 \\ \qquad\qquad\; - ax + x^2 \end{array}$$

In performing this example, we first divide a into a^2, which gives the quotient a; then multiplying the divisor $a - x$ by this quotient gives $a^2 - ax$. This subtracted from the dividend leaves $-ax + x^2$. The first term of this, viz. $-ax$ being divided by a, gives $-x$ for the second term of the quotient. Multiplying the divisor by $-x$ produces $-ax + x^2$, which, subtracted from the dividend $-ax + x^2$, leaves no remainder; $a - x$ is therefore the complete quotient.

Ex. 2.

$$\begin{array}{l} a + x)\,a^2 - x^2\,(a - x \\ \qquad\quad \underline{a^2 + ax\quad} \\ \qquad\qquad - ax - x^2 \\ \qquad\qquad \underline{- ax - x^2} \end{array}$$

Ex. 3.

$$\begin{array}{l} x + y)\,x^3 - y^3\,(x^2 - xy + y^2 - \dfrac{2y^3}{x+y} \\ \qquad\quad \underline{x^3 + x^2y\quad} \\ \qquad\qquad - x^2y - y^3 \\ \qquad\qquad \underline{- x^2y - xy^2} \\ \qquad\qquad\qquad\quad xy^2 - y^3 \\ \qquad\qquad\qquad\quad \underline{xy^2 + y^3} \\ \qquad\qquad\qquad\qquad - 2y^3 \end{array}$$

Ex. 4. $3a^2b - 2ab^3 + b^4)\,6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6\,(2ab + b^2$

$$\begin{array}{l}
\underline{6a^4b^2 - 4a^2b^4 + 2ab^5} \\
\qquad 3a^3b^3 - 2ab^5 + b^6 \\
\qquad \underline{3a^3b^2 - 2ab^5 + b^6}
\end{array}$$

Ex. 5. $x - y)\,x^5 - y^5\,(x^4 + x^3y + x^2y^2 + xy^3 + y^4$

$$\begin{array}{l}
\underline{x^5 - x^4y} \\
\quad x^4y - y^5 \\
\quad \underline{x^4y - x^3y^2} \\
\qquad x^3y^2 - y^5 \\
\qquad \underline{x^3y^2 - x^2y^3} \\
\qquad\quad x^2y^3 - y^5 \\
\qquad\quad \underline{x^2y^3 - xy^4} \\
\qquad\qquad xy^4 - y^5 \\
\qquad\qquad \underline{xy^4 - y^5}
\end{array}$$

Ex. 6. Divide $x^2 - y^2$ by $x - y$. *Ans.* $x + y$.

Ex. 7. Divide $x^2 + 2xy + y^2$ by $x + y$. *Ans.* $x + y$.

Ex. 8. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
Ans. $a^2 - 2ay + y^2$.

Ex. 9. Divide $6x^4 - 96$ by $3x - 6$.
Ans. $2x^3 + 4x^2 + 8x + 16$.

Ex. 10. Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.
Ans. $a + b$.

Ex. 11. Divide $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
by $x^2 - 2xy + y^2$. *Ans.* $x^2 - 2xy + y^2$.

Ex. 12. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$
by $2x - 3a$. *Ans.* $24x^2 - 2ax - 35a^2$.

Ex. 13. Divide $24a^4 - 192ab^3$ by $3a - 6b$.
Ans. $8a^3 + 16a^2b + 32ab^2$.

Ex. 14. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$
by $a^3 - 3a^2x + 3ax^2 - x^3$.
Ans. $a^3 + 3a^2x + 3ax^2 + x^3$.

Ex. 15. Divide $81x^4 - 18x^2 + 1$ by $9x^2 - 6x + 1$
Ans. $9x^2 + 6x + 1$

Ex. 16. Divide $a^4 - x^4$ by $a - x$.
Ans. $a^3 + a^2x + ax^2 + x^3$

Ex. 17. Divide $35x^4 - 6ax^3 - 6a^2x^2 + 27a^3x - 20a^4$ by $5x^2 - 3ax + 4a^2$.

Ans. $7x^2 + 3ax - 5a^2$.

Ex. 18. Divide $a^5 + x^5$ by $a + x$.

Ans. $a^4 - a^3x + a^2x^2 - ax^3 + x^4$.

Ex. 19. Divide $12x^4 - 5x^3y - x^2 - 2x^2y^2 + 19xy - 35$ by $4x^2 + xy - 7$. *Ans.* $3x^2 - 2xy + 5$.

Ex. 20. Divide 1 by $1 + x$.

Ans. $1 - x + x^2 - x^3 + x^4 -$ &c.

Ex. 21. Divide $1 + a$ by $1 - a$.

Ans. $1 + 2a + 2a^2 + 2a^3 +$ &c.

Ex. 22. Divide $x^4 - 4a^3x + 6a^2x^2 - 4ax^3 + a^4$, by $x^2 - 2ax + a^2$.

Ex. 23. Divide $x^7 - y^7$, by $x - y$.

Ex. 24. Divide $4x^6 - 15ax^5 + 7a^2x^4 + 14a^3x^3 - 6a^4x^2 - 4a^5x$, by $4x^2 - 3ax - 2a^2$.

Ex. 25. Divide $x^6 - a^6$, by $x^3 - 3ax^2 + 3a^2x - a^3$.

Ex. 26. Divide $21b^5 + 35b^4x + 19b^3x^2 - 18b^2x^3 - 17bx^4 - 4x^5$, by $7b^3 - 3bx^2 - x^3$.

Ex. 27. Divide $64b^6 + 16a^2b^4 - 16a^3b^3 - 8a^4b^2 + 2a^6$, by $8b^2 - 8ab + 2a^2$.

Ex. 28. Divide $1 + x$, by $1 + 2x + x^2$.

Ex. 29. Divide 1 by $1 + 1$.

Ex. 30. Divide 1 by $1 + 2$.

Ex. 31. Divide 1 by $2 + 1$.

§ **24.** The following table of factors and quotients will frequently be found useful.

1. $x^2 - y^2 = (x + y)(x - y)$.
2. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
3. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.
4. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$.

5. $\frac{x^2-y^2}{x-y}=x+y.$

6. $\frac{x^2-y^2}{x+y}=x-y.$

7. $\frac{x^3-y^3}{x-y}=x^2+xy+y^2.$

8. $\frac{x^3+y^3}{x+y}=x^2-xy+y^2.$

CHAPTER III.

ALGEBRAIC FRACTIONS.

§ **25.** Every fraction, whether it is arithmetic or algebraic, is considered as the quotient of the numerator by the denominator. The operations upon algebraic fractions are therefore performed on the same principles as in arithmetic.

The denominator in all cases indicates the number of parts into which the unit is divided, and the numerator the number of those parts expressed by the fraction.

Reduction, Addition, Subtraction, &c., are therefore performed in algebraic precisely as in vulgar fractions.

REDUCTION OF FRACTIONS.

CASE 1.

§ **26.** *To reduce fractions to their lowest terms—*

RULE.

Expunge from the numerator and denominator their common factors. The resulting fraction will be in its lowest terms; or,

Divide the numerator and denominator by their greatest common measure.

NOTE.—The determination of the common measure is often attended with some difficulty, if it is not a monomial. For rules to reduce fractions in such cases, see "A TREATISE ON ALGEBRA," by the author.

EXAMPLES.

Ex. 1. Reduce $\frac{a^2+ab}{ab-ac}$. *Ans.* $\frac{a+b}{b-c}$.

Ex. 2. Reduce $\frac{3a^3-6ab+3ab^3}{9a^3-12ab+6ab^3}$.

Ans. $\frac{a-2b+b^3}{3a-4b+2b^3}$.

Ex. 3. Reduce $\frac{7a^2-14a^2b^3}{56a^4+28a^2}$. *Ans.* $\frac{1-2b^3}{8a^2+4}$.

Ex. 4. Reduce $\frac{36a^5b^7}{54a^8b^5}$. *Ans.* $\frac{2b^2}{3a^3}$.

Ex. 5. Reduce $\frac{60a^6b^4y}{130a^5b^7y^3}$. *Ans.* $\frac{6a}{13b^3y^2}$.

Ex. 6. Reduce $\frac{a^2-y^2}{a-y}$. *Ans.* $a+y$.

CASE 2.

§**27.** *To reduce a mixed number to a fraction—*

RULE.

Multiply the integral part by the denominator, and add the numerator if the fraction is positive, but subtract if it is negative; the result will be the numerator of the required fraction.

EXAMPLES.

Ex. 1. Reduce $5a+\frac{3a-b}{7}$ to a fraction.

$$5a\times 7+3a-b=35a+3a-b=38a-b,$$

the result is $\frac{38a-b}{7}$.

Ex. 2. Reduce $3x+\frac{x-y}{4}$ to a fraction. *Ans.* $\frac{13x-y}{4}$.

Ex. 3. Reduce $7a + b - \frac{3a + 2b}{5}$ to a fraction.

Ans. $\frac{32a + 3b}{5}$.

Ex. 4. Reduce $x + y - \frac{x^2 - xy}{x - y}$ to a fraction.

Ans. $\frac{-y^2 + xy}{x - y}$.

Ex. 5. Reduce $x^4 - 3x^3y - \frac{x^5 - x^4y}{x + y}$ to a fraction.

Ans. $-\frac{x^4y + 3x^3y^2}{x + y}$.

Ex. 6. Reduce $6ax + b - \frac{6a^2x - ab}{4a}$ to a fraction.

Ans. $\frac{18a^2x + 5ab}{4a}$.

Ex. 7. Reduce $7x - 4b + \frac{5ax^2 - 4a^2b}{3ax}$ to a fraction.

Ans. $\frac{26ax^2 - 12abx - 4a^2b}{3ax}$.

Case 3.

§ **28**. *To reduce fractions to mixed quantities*—

Rule.

Divide the numerator by the denominator.

Examples.

Ex. 1. Reduce $\frac{ab - b^2}{a}$ to a mixed quantity.

Ans. $b - \frac{b^2}{a}$

Ex. 2. Reduce $\frac{by + y^2}{y}$ to an integer. *Ans.* $b + y$.

Ex. 3. Reduce $\frac{7x^2 - 4bx + b^2}{x}$ to a mixed quantity.

Ans. $7x - 4b + \frac{b^2}{x}$.

Ex. 4. Reduce $\frac{a^2 + b^2}{a + b}$ to a mixed quantity.

Ans. $a - b + \frac{2b^2}{a+b}$.

Ex. 5. Reduce $\frac{a^3 - b^3}{a + b}$ to a mixed quantity.

Ans. $a^2 - ab + b^2 - \frac{2b^3}{a+b}$.

Ex. 6. Reduce $\frac{4a^4 - 4a^3x + x^4}{4a}$ to a mixed quantity.

Ans. $a^3 - a^2x + \frac{x^4}{4a}$.

CASE 4.

§ 29. *To reduce a complex fraction to a simple one.*

RULE.

Multiply the numerator and denominator by the least common multiple of the partial fractions.

EXAMPLES.

Ex. 1. Reduce $\dfrac{a + \frac{3b}{c}}{4x - \frac{5}{d}}$ to simple fractions.

$$\frac{(a + \frac{3b}{c})\,cd}{(4x - \frac{5}{d})\,cd} = \frac{acd + 3bd}{4cdx - 5c}$$

Ex. 2. Reduce $\dfrac{3xy + \frac{4a}{x}}{7y - \frac{5x}{y}}$ to a simple fraction.

Ans. $\frac{3x^2y^2 + 4ay}{7xy^2 - 5x^2}$.

Ex. 3. Reduce $\dfrac{5x - \dfrac{3a+b}{2c}}{2c + \dfrac{4x}{3c}}$ to a simple fraction.

Ans. $\dfrac{30cx - 9a - 3b}{12c^2 + 8x}$.

Ex. 4. Reduce $\dfrac{\dfrac{3x^2+y^2}{5x} - \dfrac{2x-y^2}{3}}{4x - \dfrac{3x^2-5y^2}{3x}}$ to a simple fraction.

Ans. $\dfrac{3y^2 + 5xy^2 - x^2}{45x^2 + 25y^2}$.

Case 5.

§ **30.** *To reduce fractions to others having a common denominator.*

Rule.

Reduce the fractions to simple ones. Multiply each numerator by all the denominators but its own, for a new numerator, and multiply all the denominators together for a common denominator. Or,

When the denominators are not prime to each other, determine their least common multiple, as in arithmetic, for the common denominator. Divide each denominator into the common multiple, and multiply the quotient by the corresponding numerator; the result will be the numerator.

Examples.

Ex. 1. Reduce $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{9}{10}$ to a common denominator.

The common multiple is 40,
The numerators are

$$\frac{40}{4} \times 3 = 30, \ \frac{40}{8} \times 7 = 35, \text{ and } \frac{40}{10} \times 9 = 36,$$

consequently the fractions are

$$\frac{30}{40}, \frac{35}{40}, \text{ and } \frac{36}{40}.$$

Ex. 2. Reduce $\frac{4x}{2a^2}$, $\frac{5ax}{3ab}$, and $\frac{4x^2}{6b^2}$ to a common denominator.

The common multiple is $6a^2b^2$. Therefore,

$$\frac{6a^2b^2}{2a^2} \times 4x = 12b^2x, \frac{6a^2b^2}{3ab} \times 5ax = 10a^2bx,$$

and
$$\frac{6a^2b^2}{6b^2} \times 4x^2 = 4a^2x^2,$$

are the numerators; consequently the fractions are

$$\frac{12b^2x}{6a^2b^2}, \frac{10a^2bx}{6a^2b^2}, \text{ and } \frac{4a^2x^2}{6a^2b^2}.$$

Ex. 3. Reduce $\frac{a}{b}$ and $\frac{b}{a}$ to a common denominator

Ans. $\frac{a^2}{ab}$ and $\frac{b^2}{ab}$.

Ex. 4. Reduce $\frac{x}{y}$ and $\frac{x+y}{a}$ to a common denominator.

Ans. $\frac{ax}{ay}$ and $\frac{xy+y^2}{ay}$.

Ex. 5. Reduce $\frac{3x}{4a^2}$, $\frac{5ax}{3ab}$, and $\frac{2x^2}{5a^2}$ to a common denominator.

Ans. $\frac{45bx}{60a^2b}$, $\frac{100a^2x}{60a^2b}$, and $\frac{24bx^2}{60a^2b}$.

Ex. 6. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

Ex. 7. Reduce $\frac{x}{3y}$, $\frac{7ax}{5z}$, and $\frac{a^2-x^2}{y}$ to a common denominator.

Ans. $\frac{5xz}{15yz}$, $\frac{21axy}{15yz}$, and $\frac{15a^2z-15x^2z}{15yz}$.

Ex. 8. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a+x}{a-x}$ to a common denominator.

Ans. $\frac{7a^2-7ax}{14a-14x}$, $\frac{6ax-6x^2}{14a-14x}$, and $\frac{14a+14x}{14a-14x}$

Ex. 9. Reduce $\frac{a+x}{ab^2}$, $\frac{x}{a^3b}$, $\frac{a-x}{ax^2}$, and $\frac{a}{b^3x}$, to a common denominator.

Ans. $\frac{a^3bx^2 + a^2bx^3}{a^3b^3x^2}$, $\frac{b^2x^3}{a^3b^3x^2}$, $\frac{a^3b^3 - a^2b^3x}{a^3b^3x^2}$, and $\frac{a^4x}{a^3b^3x^2}$.

ADDITION OF FRACTIONS.

Rule.

§ 31. *Reduce the several fractions to simple ones; reduce these to others having a common denominator; then add the numerators, and place the sum over the common denominator.*

Examples.

Ex. 1. Add $\frac{2}{7}$, $\frac{4}{15}$, and $\frac{4}{21}$ together.

Here, $\frac{2}{7} + \frac{4}{15} + \frac{4}{21} =$ (Case 5, Reduction.)

$$\frac{30}{105} + \frac{28}{105} + \frac{20}{105} = \frac{78}{105}.$$

Ex. 2. Add $\frac{a^2}{4}$, $\frac{3b}{2a}$, and $\frac{5a^2+b}{6}$.

The fractions are equal to

$$\frac{3a^3}{12a}, \frac{18b}{12a}, \text{ and } \frac{10a^3 + 2ab}{12a},$$

and their sum is $\frac{13a^3 + 2ab + 18b}{12a}$.

Ex. 3. Add $\frac{x^2}{4}$, $\frac{3x^2}{5}$, and $\frac{4x^2}{10}$. *Ans.* $\frac{25x^2}{20} = \frac{5x^2}{4}$.

Ex. 4. Add $\frac{3x+2}{4}$, $\frac{4x+3}{12}$ and $\frac{5x+4}{9}$.

Ans. $\frac{59x+43}{36}$.

Ex. 5. Add $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. *Ans.* $\frac{2x^2+2y^2}{x^2-y^2}$.

Ex. 6. Add $\frac{3x-y}{3}$ and $\frac{5x}{9}$. *Ans.* $\frac{14x-3y}{9}$.

Ex. 7. Add $\frac{3x}{5}$, $\frac{4x}{15}$, and $\frac{3x+7}{3}$. *Ans.* $\frac{28x+35}{15}$.

Ex. 8. Add $a+\frac{4x^2}{b}$ and $b-\frac{3x^2}{c}$.

Ans. $a+b+\frac{4cx^2-3bx^2}{bc}$.

Ex. 9. Add $3x$, $x+\frac{3a}{4}$, and $4x-\frac{6a}{5}$.

Ans. $8x-\frac{9a}{20}$.

Ex. 10. Add $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$, and $\frac{4x}{7}$.

Ans. $\frac{68x^2+147x-35}{70x}$.

SUBTRACTION OF FRACTIONS.

Rule.

§ **32.** *Prepare the fractions as in addition; that is reduce complex fractions to simple ones, and all the fractions to a common denominator; then subtract the numerator of the subtrahend from that of the minuend, and place the difference over the common denominator.*

Examples.

Ex. 1. From $\frac{5x}{3}$ subtract $\frac{3x}{5}$. *Ans.* $\frac{16x}{15}$.

Ex. 2. From $\frac{3x-1}{4}$ subtract $\frac{2x+8}{7}$.

Ans. $\frac{13x-39}{28}$.

Ex. 3. From $\frac{5x+1}{9}$ take $\frac{3x-7}{15}$.

Ans. $\frac{16x+26}{45}$.

Ex. 4. Subtract $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$.

Ans. $\frac{4x^2-11x-5}{5x+5}$.

Ex. 5. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

Ex. 6. Subtract $\frac{1+x}{1+x+x^2}$ from $\frac{1-x}{1-x+x^2}$.

Ans. $\frac{-2x^3}{1+x^2+x^4}$.

Ex. 7. Subtract $5x-\frac{x}{c}$ from $4x+\frac{x-c}{b}$.

Ans. $\frac{bx+cx-c^2}{bc}-x$.

Ex. 8. Subtract $4x-\frac{x-4}{9}$ from $9x-\frac{3x+7}{12}$.

Ans. $5x-\frac{5x+37}{36}$.

Ex. 9. Subtract $\frac{4x^2-5}{3}$ from $\frac{8x^2-4x}{5}$.

Ans. $\frac{4x^2-12x+25}{15}$.

Ex. 10. From $\frac{5ax-4bc}{3}-\frac{2cd}{4}$ subtract $\frac{3cd}{5}-\frac{4ax+2bc}{4}$.

Ans. $\frac{160ax-50bc-66cd}{60}$.

Ex. 11. From $7ab-\frac{4x^2-5bc^2x}{2ax}$ take $4ab+\frac{3bc^2+2x}{3a}$.

Ans. $3ab-\frac{16x^2-9bc^2x}{6ax}$.

Ex. 12. From $6xy^2-2ab+\frac{7x^2-5bcx+18dx}{5b^2cx}$ take $7ab-3xy^2-\frac{4bc+11d-5x}{3b^2c}$

Ans. $9xy^2-9ab+\frac{5bcx+109dx-4x^2}{15b^2cx}$.

Ex. 13. From $3x^2y-4ay^2+\frac{5x^2-6xy+y^2}{3x}$ take $7ay^2-5x^2y-\frac{4x+2y}{5}$.

Ans. $8x^2y-11ay^2+\frac{37x^2-24xy+5y^2}{15x}$.

Ex. 14 From $18a^3-\frac{4a^2b+c^2}{4bx}$ take $14y+\frac{3a^2x-4c^2}{5x}$.

Ans. $18a^3-14y-\frac{20a^2b+5c^2+12a^2bx-16bc^2}{20bx}$.

Ex 15. From $3a^2b-5ab^2+\frac{3bc^2+8cd^2}{7bc}$ take $7ab^2+4a^2b-\frac{4bc-5d^2}{8b}$

Ans. $\frac{52bc^2+29cd^2}{56bc}-a^2b-12ab^2$.

MULTIPLICATION OF FRACTIONS.

Rule.

§ **33.** *Reduce mixed numbers to improper fractions; then multiply all the numerators together for a new numerator, and all the denominators for a new denominator.*

If any factor is found, both in a numerator and denominator, it may be cancelled before multiplying.

Examples.

Ex. 1. Multiply $\frac{5ax}{4}$ by $\frac{7a}{5}$.

Ans. $\frac{5ax}{4} \times \frac{7a}{5} = \frac{35a^2x}{20} = \frac{7a^2x}{4}$.

The 5 in the first numerator and second denominator might have been cancelled before multiplying.

Ex. 2. Multiply $\frac{x+y}{x-y}$ by $\frac{x}{y}$. *Ans.* $\frac{x^2+xy}{xy-y^2}$.

Ex. 3. Multiply $\frac{3a^2}{5b}$, $\frac{7b^2}{5x}$, and $\frac{3x}{14a}$ together.

Ans. $\frac{9ab}{50}$.

Ex. 4. Multiply $\frac{2x}{a-x}$ by $\frac{a^2-x^2}{8}$. *Ans.* $\frac{ax+x^2}{4}$.

Ex. 5. What is the product of

$\frac{3x}{5}$, $\frac{7ax}{3}$, and $\frac{14x}{5a}$. *Ans.* $\frac{98x^3}{25}$.

Ex. 6. What is the product of

$a+\frac{bx}{a}$ and $\frac{a}{x}$. *Ans.* $\frac{a^2+bx}{x}$.

Ex. 7 What is the product of

$\frac{a+x}{a}$, $\frac{a-x}{x}$, and $\frac{ax}{a^2-x^2}$. *Ans.* 1

Ex. 8. What is the product of

$$\frac{a+x}{a-x} + \frac{a-x}{a+x}, \text{ and } \frac{a^2-x^2}{a^2+x^2}. \qquad \textit{Ans.}\ 2.$$

Ex. 9. Multiply $\dfrac{a^2-x^2}{a+b}$, $\dfrac{a^2-b^2}{ax+x^2}$, and $a+\dfrac{ax}{a-x}$.

Ans. $\dfrac{a^3-a^2b}{x}$.

DIVISION OF FRACTIONS.

Rule.

§ **34.** *Reduce mixed numbers to improper fractions; then invert the divisor, and multiply the dividend by the result, as in the last case.*

Examples.

Ex. 1. Divide $\dfrac{5\,ax}{7\,b}$ by $\dfrac{3\,a}{5\,b^2}$.

Here, $\dfrac{5\,ax}{7\,b} \div \dfrac{3\,a}{5\,b^2} = \dfrac{5\,ax}{7\,b} \times \dfrac{5\,b^2}{3\,a} = \dfrac{25\,bx}{21}$ *Ans.*

Ex. 2. Divide $\dfrac{6\,x}{4}$ by $\dfrac{12\,a}{9}$. *Ans.* $\dfrac{9\,x}{8\,a}$.

Ex. 3. Divide $\dfrac{3\,a^2x}{5}$ by $\dfrac{4\,ax^2}{7}$. *Ans.* $\dfrac{21\,a}{20\,x}$.

Ex. 4. Divide $\dfrac{12\,a+6\,x}{5}$ by $\dfrac{3\,a-x}{15}$.

Ans. $\dfrac{36\,a+18\,x}{3\,a-x}$.

Ex. 5. Divide $a-\dfrac{b^2}{a}$ by $1+\dfrac{b}{a}$. *Ans.* $a-b$.

Ex. 6. Divide $\dfrac{x^4-y^4}{ab}$ by $\dfrac{x^2-y^2}{b^2}$. *Ans.* $\dfrac{bx^2+by^2}{a}$.

Ex. 7. Divide $12x^3$ by $\dfrac{(a+x)^2}{x} - a$.

Ans. $\dfrac{12x^4}{a^2+ax+x^2}$.

Ex. 8. Divide $\dfrac{2x^3}{a^3+x^3}$ by $\dfrac{x}{x+a}$. *Ans.* $\dfrac{2x}{a^2-ax+x^2}$.

Ex. 9. Divide $\dfrac{5a^2-6ab}{3}$ by $3a+\dfrac{2b-c}{8}$.

Ans. $\dfrac{40a^2-48ab}{72a+6b-3c}$

CHAPTER IV.

INVOLUTION.

§ **35.** It has been shown, in Multiplication, that to multiply together different powers of any root, we add the indices.

From this it is evident that

$a^2 \times a^2 = a^4 =$ the square of a^2.

$a^2 \times a^2 \times a^2 = a^6 =$ the cube of a^2.

$a^2 \times a^2 \times a^2 \times a^2 = a^8 =$ the fourth power of a^2, &c.

Hence, to square the power of any number, we multiply its index by 2; to cube it, multiply the index by 3, and so on.

We have, therefore, *for raising any monomial to a given power*, the following

Rule.

Multiply the indices of the several factors by that of the power to which the monomial is to be raised; the results will be the indices of the factors in the literal part. This joined to the power of the coefficient will be the power required.

EXAMPLES.

Ex. 1. What is the third power of $5\,a^2x^3$?

$$5^3 = 125,$$

third power of $\quad a^2 = a^{2\times 3} = a^6,$

" " $\quad x^3 = x^{3\times 3} = x^9,$

Hence the power required is $125\,a^6x^9$.

Ex. 2. What is the fifth power of $2\,x^2$? *Ans.* $32\,x^{10}$.

Ex. 3. Raise $3\,a^2x$ to the third power. *Ans.* $27\,a^6x^3$.

Ex. 4. Raise $-4\,a^2x^4$ to the fourth power.

Ans. $256\,a^8x^{16}$.

Ex. 5. Raise $-10\,x^3y^2$ to the third power.

Ans. $-1000\,x^9y^6$.

Ex. 6. What is the square of $-4\,a^3x^5$? *Ans.* $16\,a^6x^{10}$.

Ex. 7. What is the cube of $-3\,x^2y^3$? *Ans.* $-27\,x^6y^9$.

Ex. 8. What is the fourth power of $-3\,bc^2d$?

Ans. $81\,b^4c^8d^4$.

Ex. 9. What is the 5th power of $-2\,b^2c^3d^4$?

Ans. $-32\,b^{10}c^{15}d^{20}$.

Ex. 10. Multiply the fourth power of $4a^3x$ by the cube of $-2\,ax^3$. *Ans.* $-2048\,a^{15}x^{13}$.

Ex. 11. What is the fifth power of $\dfrac{-4\,a^3b}{3\,c^2}$?

Ans. $-\dfrac{1024\,a^{15}b^5}{243\,c^{10}}$.

Ex. 12. What is the cube of $\dfrac{7\,x^2y^3}{4\,a}$? *Ans.* $\dfrac{343\,x^6y^9}{64\,a^3}$.

§ **36.** TO RAISE POLYNOMIALS TO ANY POWER—

RULE.

Multiply the quantity continually by itself.

EXAMPLES.

Ex. 1. What is the fourth power of $a - x$?

$$\begin{array}{rl}
 & a - x \\
 & a - x \\
\hline
 & a^2 - ax \\
 & \quad - ax + x^2 \\
\hline
\text{second power,} & a^2 - 2ax + x^2 \\
 & a - x \\
\hline
 & a^3 - 2a^2x + ax^2 \\
 & \quad - a^2x + 2ax^2 - x^3 \\
\hline
\text{third power,} & a^3 - 3a^2x + 3ax^2 - x^3 \\
 & a - x \\
\hline
 & a^4 - 3a^3x + 3a^2x^2 - ax^3 \\
 & \quad - a^3x + 3a^2x^2 - 3ax^3 + x^4 \\
\hline
\text{fourth power,} & a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4
\end{array}$$

Ex. 2. What is the cube of $3a + 4b$?

$$\begin{array}{rl}
 & 3a + 4b \\
 & 3a + 4b \\
\hline
 & 9a^2 + 12ab \\
 & \qquad 12ab + 16b^2 \\
\hline
\text{square,} & 9a^2 + 24ab + 16b^2 \\
 & 3a + 4b \\
\hline
 & 27a^3 + 72a^2b + 48ab^2 \\
 & \qquad 36a^2b + 96ab^2 + 64b^3 \\
\hline
\text{cube,} & 27a^3 + 108a^2b + 144ab^2 + 64b^3
\end{array}$$

Ex. 3. What is the cube of $1 + a$?

Ans. $1 + 3a + 3a^2 + a^3$.

Ex. 4. What is the fourth power of $1 - x$?

Ans. $1 - 4x + 6x^2 - 4x^3 + x^4$.

Ex. 5. Square $2a + 3b$. *Ans.* $4a^2 + 12ab + 9b^2$.

Ex. 6. Raise $3x + 5y$ to the third power.

Ans. $27x^3 + 135x^2y + 225xy^2 + 125y^3$.

Ex. 7. Cube $a^2 - 2ay + y^2$.

Ans. $a^6 - 6a^5y + 15a^4y^2 - 20a^3y^3 + 15a^2y^4 - 6ay^5 + y^6$.

Ex. 8. Cube $2a + b - c$.

Ans. $8a^3 + 12a^2b + 6ab^2 - 12a^2c - 12abc + 6ac^2 + b^3 - 3b^2c + 3bc^2 - c^3$.

Ex. 9. Cube $x + y + z$.

Ans. $x^3 + 3x^2y + 3x^2z + 6xyz + 3xy^2 + 3xz^2 + y^3 + 3y^2z + 3yz^2 + z^3$.

Ex. 10. What is the fourth power of $\dfrac{x + 2y}{x - y}$?

$$\textit{Ans.}\quad \frac{x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4}{x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4}.$$

BINOMIAL THEOREM.

§ **37.** The preceding method of raising a polynomial to any power leads to very tedious calculations, when the index of the power is high. The following method, which is called the *Binomial Theorem*, because generally applied to expressions of two terms, greatly simplifies the operation. This theorem was discovered by Sir Isaac Newton, and was probably obtained by *Induction*, he having left no demonstration of its truth.

§ **38.** In considering this method, there are several things necessary to be kept in view.

1st. In every algebraical multiplication, the coefficients and literal portions are operated on separately.

2d. The operations on the coefficients being independent of the nature of the literal parts, will be the same, whatever those letters may be.

3d. If, therefore, we have determined the coefficients of the terms in any power, the coefficients of the same power of any other binomial will be the same, except so far as they are modified by the numerical coefficients of the separate terms of the root.

4th. The coefficients of the terms in any power of a binomial, as $x + y$, being independent of x and y, will be the same, whatever values x and y may have. These coefficients may therefore be obtained by the successive multiplication of $1 + 1$ by itself.

Performing this multiplication, we have

$$\begin{array}{rl}
 & 1 + 1 \\
 & \underline{1 + 1} \\
 & 1 + 1 \\
 & \quad\ \underline{+ 1 + 1} \\
\text{coefficient of 2d power,} & 1 + 2 + 1 \\
 & \underline{1 + 1} \\
 & 1 + 2 + 1 \\
 & \quad\ \underline{1 + 2 + 1} \\
\text{coefficient of 3d power,} & 1 + 3 + 3 + 1 \\
 & \underline{1 + 1} \\
 & 1 + 3 + 3 + 1 \\
 & \quad\ \underline{1 + 3 + 3 + 1} \\
\text{coefficient of 4th power,} & 1 + 4 + 6 + 4 + 1 \\
 & \underline{1 + 1} \\
 & 1 + 4 + 6 + 4 + 1 \\
 & \quad\ \underline{1 + 4 + 6 + 4 + 1} \\
\text{coefficient of 5th power,} & 1 + 5 + 10 + 10 + 5 + 1 \\
 & \underline{1 + 1} \\
 & 1 + 5 + 10 + 10 + 5 + 1 \\
 & \quad\ \underline{1 + 5 + 10 + 10 + 5 + 1} \\
\text{coefficient of 6th power,} & 1 + 6 + 15 + 20 + 15 + 6 + 1 \\
 & \&c
\end{array}$$

§ **39.** By examining these series, we may observe,

1st. That the terms are the same, whichever end we commence with.

2d. That the second term is always the same as the index of the power; and

3d. That the remaining ones may be obtained thus, (taking the sixth power as an example:)

$$\frac{6 \times 5}{2} = 15, \quad \frac{15 \times 4}{3} = 20, \quad \frac{20 \times 3}{4} = 15, \quad \frac{15 \times 2}{5} = 6,$$

and $$\frac{6 \times 1}{6} = 1.$$

§ **40.** We may likewise remark, that in raising a binomial such as $x + y$, to any power, the powers of each letter will

be independent of the other. Thus the powers of x, in $(x+y)^4$ and $(x+1)^4$, will be the same, as will also be the powers of y, in $(x+y)^4$ and $(1+y)^4$.

If then we multiply $x+1$ and $1+y$ by themselves, we shall obtain

$$
\begin{array}{ll}
 & x + 1 \\
 & x + 1 \\
\cline{2-2}
 & x^2 + x \\
 & \qquad x + 1 \\
\cline{2-2}
\text{square,} & x^2 + 2x + 1 \\
 & x + 1 \\
\cline{2-2}
 & x^3 + 2x^2 + x \\
 & \qquad x^2 + 2x + 1 \\
\cline{2-2}
\text{cube,} & x^3 + 3x^2 + 3x + 1 \\
 & x + 1 \\
\cline{2-2}
 & x^4 + 3x^3 + 3x^2 + x \\
 & \qquad x^3 + 3x^2 + 3x + 1 \\
\cline{2-2}
\text{4th power,} & x^4 + 4x^3 + 6x^2 + 4x + 1
\end{array}
$$

$$
\begin{array}{ll}
 & 1 + y \\
 & 1 + y \\
\cline{2-2}
 & 1 + y \\
 & \quad y + y^2 \\
\cline{2-2}
\text{square,} & 1 + 2y + y^2 \\
 & 1 + y \\
\cline{2-2}
 & 1 + 2y + y^2 \\
 & \quad y + 2y^2 + y^3 \\
\cline{2-2}
\text{cube,} & 1 + 3y + 3y^2 + y^3 \\
 & 1 + y \\
\cline{2-2}
 & 1 + 3y + 3y^2 + y^3 \\
 & \quad y + 3y^2 + 3y^3 + y^4 \\
\cline{2-2}
\text{4th power,} & 1 + 4y + 6y^2 + 4y^3 + y^4
\end{array}
$$

§ **41.** By examining these results, we perceive that the powers of x begin with that of the binomial, and regularly descend, the last term being 1; while in development of the powers of $1+y$, the first term is 1, and the powers of y in the succeeding terms regularly ascend by unity.

Therefore for the development of the power of a binomial we have the following

RULE FOR OBTAINING THE INDICES.

The indices of the first term of the root will begin in the first term of the development, with that of the binomial, and regularly descend by unity; and those of the second term of the root will commence in the second term of the development with a unit, and regularly ascend by unity until they reach that of the binomial, thus:

The complete literal portion of the 4th power of $x + y$ is

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4,$$

and if the coefficients of the 4th power, viz. 1, 4, 6, 4, 1, be applied to these terms, we shall have

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

for the complete 4th power of the binomial.

So of the 6th power the literal portion is

$$x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6.$$

§ **42.** Now calling x the leading quantity in this development, and comparing the indices of the different powers of x with the several factors used in obtaining the coefficients in § 39, we readily derive the following

RULE FOR OBTAINING THE COEFFICIENTS.

1. *The coefficient of the first term is unity.*

2. *The coefficient of the second term is the index of the power of the binomial.*

3. *The succeeding coefficients are determined by multiplying the coefficient of the preceding term by the index of the leading quantity in that term, and dividing the product by that of the other quantity increased by unity.*

If one of the terms of the binomial be negative, all the odd powers of that term in the power will likewise be negative.

EXAMPLES.

Ex. 1. Raise $a - b$ to the 5th power.

Here the literal portion is

$$a^5\ a^4b\ a^3b^2\ a^2b^3\ ab^4\ b^5$$

The coefficients are

$$1, 5, \frac{5 \times 4}{2} = 10, \frac{10 \times 3}{3} = 10, \frac{10 \times 2}{4} = 5, \frac{5 \times 1}{5} = 1.$$

The complete power is

$$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

The whole process may be neatly arranged thus:

$$\begin{array}{ccccccccccc} a^5 & - & 5a^4b & + & 10a^3b^2 & - & 10a^2b^3 & + & 5ab^4 & - & b^5. \\ & & 4 & & 3 & & 2 & & 1 & & \\ & & 2)20 & & 3)30 & & 4)20 & & 5)5 & & \\ & & 10 & & 10 & & 5 & & 1 & & \end{array}$$

It is manifestly unnecessary to pursue the calculation for the coefficients beyond the middle term, since beyond this point they recur in a reverse order.

Ex. 2. Raise $x + y$ to the 6th power.

Ans. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3$, &c.

Ex. 3. Raise $a - b$ to the 7th power.

Ans. $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4$, &c.

Ex. 4. Raise $x - y$ to the 9th power.

Ans. $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 -$ &c.

Ex. 5. Raise $2a + 3b$ to the 4th power.

In performing this example, the coefficients must be determined as though neither of the terms of the binomial had a coefficient. The several terms of the development may afterward be reduced. Thus,

$$\begin{array}{ccccccccc} (2a+3b)^4 = (2a)^4 & + & 4(2a)^3\,3b & + & 6(2a)^2(3b)^2 & + & 4(2a)\,(3b)^3 & + & (3b)^4 \\ & & 3 & & 2 & & 1 & & \\ & & 2)12 & & 3)12 & & 4)4 & & \\ & & 6 & & 4 & & 1 & & \end{array}$$

$$= 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4.$$

Ex. 6. Raise $x + 3$ to the 5th power,

Here,

$$(x+3)^5 = x^5 + 5x^4.3 + 10x^3.3^2 + 10x^2.3^3 + 5x.3^4 + 3^5$$
$$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243.$$

Ex. 7. What is the square of $3a - 4b$?

Ans. $9a^2 - 24ab + 16b^2$.

Ex. 8. What is the square of $2b + 5x$?
Ans. $4b^2 + 20bx + 25x^2$.

Ex. 9. What is the cube of $3a - x$?
Ans. $27a^3 - 27a^2x + 9ax^2 - x^3$.

Ex. 10. What is the cube of $5 + 2x$?
Ans. $125 + 150x + 60x^2 + 8x^3$.

Ex. 11. What is the 4th power of $a - 5x$?
Ans. $a^4 - 20a^3x + 150a^2x^2 - 500ax^3 + 625x^4$.

Ex. 12. What is the 4th power of $3a - 2b$?
Ans. $81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$.

Ex. 13. What is the 7th power of $2 - y$?
Ans. $128 - 448y + 672y^2 - 560y^3 + 280y^4 - 84y^5 + 14y^6 - y^7$.

EVOLUTION.

§**43.** The square of any number is the product arising from multiplying that number by itself.

The *square root* of any number is the number whose square is the given number.

Thus, $4 \times 4 = 16$, is the square of 4, and 4 is the square root of 16.

§**44.** The square root of any perfect integral square number less than 100, can only be found by inspection. Thus, if the square root of 81 were required, we know that $9 \times 9 = 81$, and therefore 9 is the square root of 81.

§**45.** The mode of extracting the square root of numbers has been given in Arithmetic. That of obtaining the root of monomial algebraic quantities is readily derived from the rule for Involution.

§**46.** We have learned that to raise an algebraic monomial to any power, we multiply its index by that of the index of the given power.

Consequently, to extract the root, we divide the index by 2 for the square root, by 3 for the cube root, and so on.

§ **47.** Since $+a \times a = a^2$, and $-a \times -a = a^2$, the square root of a^2 is either $+a$ or $-a$, this result is written $\pm a$. Similarly the fourth, sixth, &c. roots of a quantity are either $+$ or $-$.

From the above it is plain that an even root of a negative quantity is impossible.

$\sqrt{-a^2}$, for example, is neither $+a$ or $-a$, for either of these squared, produces a^2.

The even roots of negative quantities, not having any value, are called *Imaginary Quantities.*

EXAMPLES.

Ex. 1. What is the square root of a^4? *Ans.* $\pm a^2$.

Ex. 2. What is the cube root of x^6? *Ans.* x^2.

Ex. 3. What is the square root of a^2y^2? *Ans.* $\pm ay$.

Ex. 4. What is the cube root of x^9y^6? *Ans.* x^3y^2.

Ex. 5. What is the square root of $9x^8$? *Ans.* $\pm 3x^4$.

Ex. 6. What is the 4th root of $16a^8x^{12}$?
Ans. $\pm 2a^2x^3$.

Ex. 7. What is the cube root of $-125x^6y^9z^3$?
Ans. $-5x^2y^3z$.

Ex. 8. What is the 5th root of $32x^{10}y^{15}$? *Ans.* $2x^2y^3$.

Ex. 9. What is the 10th root of $a^{10}x^{20}y^{30}$?
Ans. $\pm ax^2y^3$.

Ex. 10. What is the square root of $784a^4x^8y^{12}$?
Ans. $\pm 28a^2x^4y^6$.

SQUARE ROOT OF POLYNOMIALS.

§ **48.** The square of $a + x$ being $a^2 + 2ax + x^2$, the square root of $a^2 + 2ax + x^2$ must be $a + x$.

The first term a of the root is the root of a^2 the first term of the power. This term being taken from the power leaves $2ax + x^2$ for the remainder; this divided by $2a + x$ gives x, the remaining term of the root.

From this investigation we may derive the following

Rule.

1st. *Arrange the terms, beginning with the highest power of one of the letters, and regularly descending; or with the lowest, and regularly ascending.*

2d. *Take the root of the first term for the first term of the root.*

3d. *Bring down so many of the remaining terms of the power as are required for a dividend. Divide the first term of the dividend by twice the ascertained root, and place the quotient in the root and likewise in the divisor.*

4th. *Multiply the divisor thus increased by the term of the root just determined, and subtract the product from the dividend, and so proceed until all the terms have been brought down.*

Examples.

Ex. 1. What is the square root of

$$4a^2 - 12ax + 9x^2?$$

$$\begin{array}{r|l} & 4a^2 - 12ax + 9x^2 \,(2a - 3x \\ & 4a^2 \\ \hline 4a - 3x\,) & -12ax + 9x^2 \\ & -12ax + 9x^2 \\ \hline \end{array}$$

Ex. 2. What is the square root of

$$9x^4 - 12x^3 + 16x^2 - 8x + 4?$$

$$\begin{array}{r|l} & 9x^4 - 12x^3 + 16x^2 - 8x + 4\,(3x^2 - 2x + 2 \\ & 9x^4 \\ \hline 6x^2 - 2x\,) & -12x^3 + 16x^2 \\ & -12x^3 + 4x^2 \\ \hline 6x^2 - 4x + 2\,) & 12x^2 - 8x + 4 \\ & 12x^2 - 8x + 4 \\ \hline \end{array}$$

Ex. 3. What is the square root of

$a^2 + 2ax + x^2$? *Ans.* $a + x$.

Ex. 4. What is the square root of

$9a^2 + 6ax + x^2$? *Ans.* $3a + x$.

Ex. 5. What is the square root of
$a^4 - 24a^2b + 144b^2$? *Ans.* $a^2 - 12b$.

Ex. 6. What is the square root of
$4x^4 - 16x^3 + 24x^2 - 16x + 4$?
Ans. $2x^2 - 4x + 2$.

Ex. 7. What is the square root of
$a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$?
Ans. $a^2 + 2ax + x^2$.

Ex. 8. What is the square root of
$16a^4b^4 - 40a^3b^3 + 25a^2b^2$? *Ans.* $4a^2b^2 - 5ab$.

Ex. 9. What is the square root of
$\dfrac{16x^2 - 8ax + a^2}{9b^2 + 12by + 4y^2}$? *Ans.* $\dfrac{4x - a}{3b + 2y}$.

Ex. 10. What is the square root of
$\dfrac{25a^4 - 30a^2b^2 + 9b^4}{16b^2 - 24ab + 9a^2}$? *Ans.* $\dfrac{5a^2 - 3b^2}{4b - 3a}$.

Ex. 11. What is the square root of $a^2 + 1$?

$$a^2 + 1 \left(a + \frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5} - \frac{5}{128a^7}, \text{ \&c.}\right.$$
$$a^2$$
$$2a + \frac{1}{2a}\Big)\ 1$$
$$1 + \frac{1}{4a^2}$$
$$2a + \frac{1}{a} - \frac{1}{8a^3}\Big) - \frac{1}{4a^2}$$
$$-\frac{1}{4a^2} - \frac{1}{8a^4} + \frac{1}{64a^6}$$
$$2a + \frac{1}{a} - \frac{1}{4a^3} + \frac{1}{16a^5}\Big)\ \frac{1}{8a^4} - \frac{1}{64a^6}$$
$$\frac{1}{8a^4} + \frac{1}{16a^6} - \frac{1}{64a^8} + \frac{1}{256a^{10}}$$
$$2a + \frac{1}{a} - \frac{1}{4a^3} + \frac{1}{8a^5}\Big) - \frac{5}{64a^6} + \frac{1}{64a^8} - \frac{1}{256a^{10}}.$$

Ex. 12. What is the square root of

$$1 + a^2?$$

Ans. $1 + \frac{a^2}{2} - \frac{a^4}{8} + \frac{a^6}{16} - \frac{5\,a^8}{128} + \text{\&c.}$

Ex. 13. What is the square root of

$$a^2 + b?$$

Ans. $a + \frac{b}{2a} - \frac{b^2}{8a^3} + \frac{b^3}{16\,a^5} - \frac{5\,b^4}{128\,a^7} + \text{\&c.}$

Ex. 14. What is the square root of

$$4\,a^2 - 3\,b?$$

Ans. $2\,a - \frac{3\,b}{4\,a} - \frac{9\,b^2}{64\,a^3} - \frac{27\,b^3}{512\,a^5} - \text{\&c.}$

Ex. 15. What is the square root of

$$1 + 2\,a?$$

Ans. $1 + a - \frac{a^2}{2} + \frac{a^3}{2} - \frac{5\,a^4}{8} + \frac{7\,a^5}{8} - \text{\&c.}$

Ex. 16. Extract the square root of

$$4\,x^2 - 2\,x + 1.$$

Ans. $2\,x - \frac{1}{2} + \frac{3}{16x} + \frac{3}{64\,x^2}$, &c.

Ex. 17. Extract the square root of

$$a^{10} - 10\,a^9b + 45\,a^8b^2 - 120\,a^7b^3 + 210\,a^6b^4 - 252\,a^5b^5 + 210\,a^4b^6 - 120\,a^3b^7 + 45\,a^2b^8 - 10\,ab^9 + b^{10}.$$

Ans. $a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5.$

CHAPTER V.

SIMPLE EQUATIONS.

§ **49.** An expression of *equality* between two quantities is called an *equation.*

§ **50.** The quantities considered equal are the members or sides of the equation. Thus,

$$4x + 2 = 3x + 11$$

is an equation, $4x + 2$ being the left hand or first member, and $3x + 11$ the right hand or second member.

§ **51.** Equations are characterized by the highest power of the unknown found in them. Thus,

$$3x + 11 = 4x + 2$$

is an equation of the *first degree*, or a *simple equation;*

$$4x^2 - 5 = 2x + 3$$

is an equation of the *second degree*, or a *quadratic equation.*

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

§ **52.** Every equation requiring to be solved contains known and unknown quantities, and the object to be obtained in the solution is the determination of the values of the latter.

§ **53.** The following principles lead to the solution of equations.

1st. If the same number or quantity be added to or subtracted from both members of the equation, the equality will not be destroyed. Thus if we have

$$4x - 20 = 3x + 56,$$

the addition of 20 to both members gives

$$4x = 3x + 56 + 20$$

or, $$4x = 3x + 76$$

The subtraction of $3x$ from both gives

$$x = 76.$$

2d. Both members may be multiplied or divided by the same number without destroying the equality.

Thus if both members of the equation

$$4x - 10 = 2x + 8$$

be multiplied by 4, we shall have

$$16x - 40 = 8x + 32.$$

TRANSPOSITION.

§ **54.** From the first of the above principles it follows that by changing the sign of any quantity it may be transposed from one member of an equation to the other, without destroying the equality.

Thus, in the equation

$$4x - 20 = 3x + 56,$$

the addition of 20 to both members gives

$$4x = 3x + 56 + 20 = 3x + 76.$$

by which operation it will be perceived that $+20$ has been substituted in the second member of the equation, for -20, which appeared in the first member.

The subtraction of $3x$ from both members gives

$$4x - 3x = 76. \quad \text{Hence, } x = 76.$$

From the consideration of the principle above stated, we derive the following

RULE.

To transpose a quantity, change its sign and remove it from that member of the equation in which it is found, to the other.

NOTE 1.—To solve an equation by transposition, collect all the terms containing the unknown quantity to the left-hand member of the equation, and the known terms to the other.

NOTE 2.—Should the first member, after the equation has been reduced be negative, change the signs of all the terms in both members. Thus,

Let $2x + 29 = 3x + 11$. We have, by transposition,

$$2x - 3x = 11 - 29$$

or $-x = -18$, or, by transposition, $18 = x$

consequently, $x = 18$.

EXAMPLES.

Ex. 1. Solve the equation

$$5x - 40 = 4x + 10 \quad \text{by transposition.}$$

The operation will stand

$$5x - 4x = 10 + 40,$$

or reducing, $x = 50.$

Ex. 2. Solve by transposition

$7x + 40 = 6x + 90.$ *Ans.* $x = 50.$

Ex. 3. Solve by transposition

$$3x + 10 + 5x = 19 + 7x + 12.$$

The operation is

$$3x + 5x - 7x = 19 + 12 - 10,$$

or, $x = 21.$

Ex. 4. Solve by transposition

$7x - 4 + 3x = 13x + 9 - 4x.$ *Ans.* $x = 13.$

Ex. 5. Transpose and reduce

$9x - 7 + 4x + 8 = 3x + 5 + 7x + 15.$

Ans. $3x = 19.$

Ex. 6. Transpose and reduce

$14 + 8x - 3 = 6x + 15.$ *Ans.* $2x = 4.$

Ex. 7. Transpose and reduce

$25x + 16 - 3x = 27x + 28 - 8x.$

Ans. $3x = 12.$

Ex. 8. Transpose and reduce

$36 = 48 - 3x + 12.$ *Ans.* $3x = 24.$

CLEARING OF FRACTIONS.

§ **55.** The second principle enables us to clear an equation of fractions. For this purpose we have the following

RULE.

Multiply each member of the equation by the least common multiple of the denominators, or

When the denominators are prime to each other, multiply each numerator by all the denominators except its own.

NOTE.—If any of the fractions are negative, the signs of all the terms produced by multiplying those fractions must be changed. (See Ex. 3.)

EXAMPLES.

Ex. 1. Clear the equation

$$\frac{x}{2} + 15 = \frac{7x}{4} - \frac{3}{2} \text{ of fractions.}$$

Multiplying by 4 it becomes

$$2x + 60 = 7x - 6.$$

Ex. 2. Clear the equation

$$\frac{3x+5}{4} + \frac{3}{2} = \frac{2x+7}{6} - 8 \text{ of fractions.}$$

Multiplying by 12, the least common multiple of the denominators, we obtain

$$9x + 15 + 18 = 4x + 14 - 96.$$

Ex. 3. Clear of fractions the equation

$$\frac{7x-6}{5} - \frac{4x-3}{10} = \frac{2x+9}{15}.$$

Multiplying by 30 we obtain

$$42x - 36 - 12x + 9 = 4x + 18.$$

In this example the fraction $\frac{4x-3}{10}$ being multipled by 30, gives $12x - 9$, and as this fraction is subtractive, the signs of the result must be changed in accordance with the general rule for subtraction. It must therefore be written $-12x + 9$, as above. From this we perceive that the signs of every term of the numerator of a negative fraction must be changed, when the equation is cleared of fractions

Ex. 4. Clear the equation

$$\frac{5x}{6} + \frac{3}{4} = \frac{7x}{12} + 8 \text{ of fractions.}$$

Ans. $10x + 9 = 7x + 96.$

Ex. 5. Clear the equation

$$\frac{7x+2}{3} - \frac{3}{4} = \frac{2x+3}{6} + 5 \text{ of fractions.}$$

Ans. $28x + 8 - 9 = 4x + 6 + 60$

Ex. 6. Clear the equation

$$\frac{5x-3}{4}-\frac{3x}{5}=\frac{2x-8}{10}+2$$

of fractions, transpose and reduce as in last rule.

Ans. $9x=39$.

Ex. 7. Clear the following equation, and transpose the terms.

$$\frac{4x-3}{7}-\frac{2x-8}{4}=\frac{x}{14}-\frac{4+3x}{7}+4.$$

Ans. $12x=52$.

Ex. 8. Given the equation

$$\frac{7x-5}{15}-\frac{4x+3}{5}=\frac{7+3x}{4}-4x.$$

Ans. $175x=161$.

§ **56.** In the last three examples the solution might have been carried further by dividing by the coefficient of x. We should thus have obtained the value of the unknown.

Thus, in Ex. 6, we have $9x = 39$,

whence, dividing by 9, $x = \frac{39}{9} = 4\frac{1}{3}$.

So in Ex. 7, from $12x = 52$ we obtain

$$x = \frac{52}{12} = 4\frac{1}{3},$$

and in Ex. 8, from $175x = 161$ we obtain

$$x = \frac{161}{175} = \frac{23}{25}.$$

GENERAL RULE FOR SOLVING SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

§ **57.** *Clear the equation of fractions, transpose all the terms containing the unknown to the left hand member, collect the several terms together, and divide by the coefficient with which the unknown is affected in the resulting equation.*

NOTE.—When the same quantity is found in both members, affected by the same sign, it may be stricken out.

Examples.

Ex. 1. Given $3x - 2 + 27 = 34$, to find x.

Ans. $x = 3$.

Ex. 2. Given $4y + 8 = 2y + 16$, to find y.

Ans. $y = 4$.

Ex. 3. Given $\frac{3x}{4} - 7 = \frac{2x}{3} - 6$, to find x.

Ans. $x = 12$.

Ex. 4. Given $x + \frac{1}{3}x + 8 = \frac{1}{4}x + 11 - \frac{1}{2}$.

Ans. $x = 2\frac{4}{13}$.

Ex. 5. Given $3x - \frac{1}{3}x + 1 = \frac{7}{5}x + 2$, to find x.

Ans. $x = \frac{15}{19}$

Ex. 6. Given $\frac{x-5}{2} + \frac{x}{5} = 17 - \frac{x+10}{5}$, to find x.

Ans. $x = 19\frac{4}{9}$.

Ex. 7. Given $\frac{x+3}{2} + \frac{x}{3} = 4 - \frac{x-5}{4}$, to find x.

Ans. $x = 3\frac{6}{13}$.

Ex. 8. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 7 - \frac{x}{5}$, to find x.

Ans. $x = 5\frac{5}{71}$.

Ex. 9. Given $\frac{x}{a} + \frac{x-2}{3} = \frac{2x+5}{2a}$, to find x.

Ans. $x = \frac{4a+15}{2a}$.

Ex. 10. Given $3x - \frac{4x-10}{13} = \frac{22+4x}{11} + 50$, to find x.

Ans. $x = 22$

Ex. 11. Given $\frac{x}{7} - \frac{4x}{9} + \frac{3-x}{5} = 11\frac{1}{15} - x$, to find x.

Ans $x = 21$.

Ex. 12. Given $\frac{3x}{5} - \frac{x+5}{4} = \frac{16-3x}{10} + 8\frac{1}{5}$, to find x.

Ans. $x = 17$.

Ex. 13. Given $\frac{2x+7}{5} - \frac{4}{7} = \frac{24+2x}{5} - \frac{10x+69}{5x}$, to find x.

Ans. $x = 7$.

Ex. 14. Given $5x + \frac{7x+9}{2x+3} = 9 + \frac{10x^2-18}{2x+3}$, to find the value of x.

Ans. $x = 0$.

Ex. 15. Given $4x - \frac{3x-49}{7} + \frac{5-7x}{14} = 20 - \frac{16+5x}{21} + \frac{8x-1}{3}$

Ans. $x = 5$.

Ex. 16. Given $7x - \frac{4x+9}{3x-7} = 27 + \frac{21x-5}{3}$

Ans. $x = 2\frac{5}{48}$.

Ex. 17. Given $\frac{3x-5}{a} - \frac{2x+7}{3a} = \frac{9-4x}{2a} - \frac{3+7x}{6}$

Ans. $x = \frac{71-3a}{26+7a}$.

EXAMPLES PRODUCING EQUATIONS OF THE FIRST DEGREE, INVOLVING ONE UNKNOWN QUANTITY.

§**58.** In the solution of a question by Algebra, it is necessary first to study well the conditions of the problem, and then endeavour, by assuming some letter to represent the unknown quantity, to perform upon it the operations indicated by the question. By this means an equation may be formed, and the value of the unknown determined by the methods already laid down.

§**59.** The operations to be performed in *putting the question into an equation*, depend so much on the nature of the problem, that no rules of general application can be given. It not unfrequently happens that the conditions indicate at once the nature of the equation to be formed, as in the following example.

To find two numbers whose sum is 20, and difference 8.

In this case, if x represent the less, $x + 8$ must represent the greater, and as their sum is 20, we have

$$x + x + 8 = 20,$$

whence $$x = 6, \text{ and } x + 8 = 14.$$

Sometimes, on the contrary, the conditions are so involved that some ingenuity is necessary, not merely in selecting the quantity to be considered as the primary unknown, but also in forming an equation with this unknown and the *given quantities.* Thus,

A. can perform a piece of work in 12 days, and B. can perform the same in 15 days; in what time will they finish it, working together?

The conditions on which this problem must be solved are evidently that A. and B. jointly, will do as much work as they will do working separately.

Let then x represent the time required for them, working together, to finish the work.

Then, A. will do $\frac{1}{12}$ of it in 1 day,

B. will do $\frac{1}{15}$ of it in 1 day,

and A and B. will do $\frac{1}{x}$ in the same time.

Consequently we shall have

$$\frac{1}{12}+\frac{1}{15}=\frac{1}{x},$$

whence, clearing of fractions,

$$5x+4x=60,$$

or, $$x=6\tfrac{2}{3} \text{ days.}$$

EXAMPLES.

Ex. 1. Required to divide a line of 21 yards in length into two parts, such that the one shall be twice as long as the other.

Let x represent the shorter part, then $2x$ will represent the longer, and by the conditions we shall have

$$x+2x=3x=21,$$

whence and $$\left.\begin{aligned} x &= 7 \\ 2x &= 14 \end{aligned}\right\} \text{the required portions.}$$

Ex. 2. An estate of \$30,000 is to be divided so that the eldest brother is to have \$3000 more than the second, and the second twice as much as the third. Required their portions.

Here, if x represent the youngest son's share, $2x$ and $2x+3000$ will be the shares of his brothers; and as the sum of the shares must make the whole estate, we shall have $$x+2x+2x+3000=30000,$$

whence $$5x=27000,$$

and $$x=5400, \text{ the youngest son's portion.}$$

Consequently, and $$\left.\begin{aligned} 2x &= 10800 \\ 2x+3000 &= 13800 \end{aligned}\right\} \text{will be the shares of his brothers.}$$

Ex. 3. Required to find a number whose third part exceeds its fourth part by 20.

Here, if x be the number, we shall have

$$\frac{1}{3}x - \frac{1}{4}x = 20,$$

whence, clearing of fractions,

$$4x - 3x = 240,$$

or, $x = 240$, the number required.

Ex. 4. A criminal escaping from prison travels 3 days before his route is discovered; he walks 10 hours a day at the rate of 3 miles per hour. An officer then starts in pursuit, travelling 8 hours a day at the rate of 7 miles per hour. How far will the man have gone before he is overtaken?

Representing the time required by the officer by x, the time the criminal will have journeyed will be $x + 3$.

Now since the officer travels 56 miles per day, and the man 30 miles,

$$56x \text{ and } 30x + 90$$

will represent the number of miles they travel respectively, and as these must represent the same distance, we shall have the equation

$$56x = 30x + 90$$

whence $x = 3\frac{6}{13}$, the number of days the officer travels,

and $56x = 193\frac{11}{13}$, the number of miles travelled.

Ex. 5. What number is that, to the double of which, if 16 be added, the sum shall be 66? *Ans.* 25.

Ex. 6. There is a certain number, such that if we take 17 from three times the number, the remainder shall be equal to the number itself increased by 5. *Ans.* 11.

Ex. 7. A man has three times as many cows as oxen, and

four times as many sheep as cows. Now, the number of all being 48, how many of each kind had he?

Ans. 3 oxen, 9 cows, and 36 sheep.

Ex. 8. A boy has five times as many apples as pears, and four times as many pears as peaches. Now, as he has 125 in all, what number has he of each kind?

Ans. 100 apples, 20 pears, 5 peaches.

Ex. 9. John bought some oranges, lemons, and peaches, for 96 cents, having laid out four times as much in oranges as in lemons, and three times as much for peaches as for lemons. What did he lay out for each?

Ans. 48 cts. for oranges, 36 cts. for peaches, and 12 cts. for lemons.

Ex. 10. A post is half in the water, one-third in the air, and 4 feet in the mud. What is its length?

Ans. 24 ft.

Ex. 11. Five times a certain number increased by 15 is equal to the third of it multiplied by 20. What is the number? *Ans.* 9.

Ex. 12. A certain number increased by 18, and then diminished by one-half the number, is equal to 120. What is the number? *Ans.* 204.

Ex. 13. One-half the trees in an orchard bear apples; one-fourth bear peaches, and one-fifth bear pears, and 40 bear cherries. How many are there in all? *Ans.* 800.

Ex. 14. One-fourth of the pupils in a school are over 16 years of age; one-third are between 14 and 16; now if there were 44 under 14, the school would be half as large again as it is. How many pupils are there? *Ans.* 48.

Ex. 15. A person finds that he had spent one-fourth of his life before going to school, one-sixth of it at a primary

school, one-eighth at an academy, and one-fourth of it at college. Now, as it has been 5 years since he left college, how old is he? *Ans.* 24 years.

Ex. 16. I spent one-sixth of my money for clothes, one-fifth for board, and one-fourth in travelling. Now, as I have \$345 left, how much had I at first? *Ans.* \$900.

Ex. 17. A father left his estate so that his widow should have one-third; his eldest son \$5000 more than the second, and the second \$3000 more than the third. What was the estate, and each one's share, that of the youngest being half what the widow received?

Ans. Estate = \$66000; widow's share, \$22000; sons', \$19000, \$14000, and \$11000.

Ex. 18. A father left his property among his three sons, as follows: John was to have \$5000 less than half of it, Robert was to have \$1000 more than the fourth of it, and Thomas \$7000 more than the fifth. What was the value of the whole estate? *Ans.* \$60000.

Ex. 19. A person spent one-sixth of his life in childhood, three-tenths of it at school, after 2 years he was married. Now, as one-half his life was spent in a married state, what was his age? *Ans.* 60 years.

Ex. 20. An estate is to be divided so that Thomas shall receive twice as much as John, wanting \$1000, and William have three times as much as Thomas, wanting \$1800. When the settlement was made, it was found that William's share was equal to three times John's. What was the whole estate? *Ans.* \$8600.

Ex. 21. A. and B. began trade with equal stocks; in the first year, A. trebled his stock and had \$27 to spare; B. doubled his and had \$153 to spare. Now the amount of their gains being five times the original stock of either, what was that stock? *Ans.* \$90.

Ex. 22. In a mixture of wheat and rye, one-half the

whole + 25 bushels was wheat, and one-third of the whole — 5, was rye. How many bushels were there of each?

Ans. 85 bu. of wheat, and 35 bu. of rye.

Ex. 23. A gentleman purchased a carriage and pair of horses, with the harness, for \$1500; the horses cost five times as much as the harness, and the carriage \$300 more than them both. What did he give for each?

Ans. Harness, \$100; horses, \$500; carriage, \$900.

Ex. 24. After a battle it was found that half the army + 3600 men were fit for action, one-eighth + 600 being wounded, and the rest, which was one-fifth of the whole, being slain or missing. How many men were there at first?

Ans. 24000.

Ex. 25. At a certain election the successful candidate had 1350 majority. Now, as the whole number of votes polled was 12696, required the number of votes each received.

Ans. 7023 and 5673.

Ex. 26. On dividing a tract of land containing 1115 acres, it was found that James received 125 acres more than Thomas, and Thomas 130 acres less than twice what William received. How much had each?

Ans. James 495 A., Thomas 370 A., and William 250 A.

Ex. 27. In a mixture of corn and oats, one-third of the whole plus 25 bushels was corn, and one-half less 7 bushels was oats. How much was there in all? *Ans.* 108 bushels.

Ex. 28. A teacher, being asked the number of his pupils, replied, that if he had as many more, half as many more, and five and a half, he should have 123. What number had he?

Ans. 47.

Ex. 29. What number is that whose third, fourth, fifth, and sixth parts will make the number itself diminished by 3

Ans. 60.

Ex. 30. Five times a certain number diminished by 7 is equal to three times that number increased by 12. What is the number? *Ans.* $9\frac{1}{2}$.

Ex. 31. A gentleman, wishing to buy cloth for a suit of clothes, found that if he bought blue cloth at \$6 per yard, he would need \$4 more than he had about him to pay for it; but if he bought brown cloth at \$4 per yard he would have \$6.50 to spare. How many yards did he require? *Ans.* $5\frac{1}{4}$.

Ex. 32. A man wishing to make a present to some boys, found that it would require 50 cents more than he had to give each 10 cents; he therefore gave each 8 cents, and had 20 cents remaining. How many boys were there? *Ans.* 35.

Ex. 33. A. and B. play; after A. loses \$10, he wants \$8 in order to have as much as B. How much has each, supposing A. had twice as much as B. at the commencement? *Ans.* A. had \$24, and B. \$12.

Ex. 34. A cistern is supplied with 3 cocks. The first would empty it in 40 minutes, the second in 50 minutes, and the third in 60 minutes. In what time would it be emptied if all were left running? *Ans.* $16\frac{8}{37}$ minutes.

Ex. 35. The pumps that supply the reservoir of a certain city, would fill it with water in 18 hours. There are two mains leading to the town, one of which would empty it in 20 hours, and the other in 24 hours. Now if the pumps be set at work and both mains left open, in what time would the reservoir be emptied? *Ans.* $27\frac{9}{13}$ hours.

Ex. 36. A cistern has three cocks by which it may be filled, and two for discharging. By the first alone it will be filled in 5 hours; the second alone would fill it in 6 hours; the third, in $4\frac{1}{2}$ hours. The discharging cocks would empty it in 3 and 4 hours respectively. Now, how long will it require to fill the cistern, if all the cocks be left open, the contents being 500 gallons? *Ans.* 180 hours.

Ex. 37. A. can do a piece of work in 12 days, and B. can do it in 20 days. In what time can they do it, working together? *Ans.* $7\frac{1}{2}$ days.

Ex. 38. A can do a piece of work in a days, B. in b days, and C. in c days. In what time can they perform it, working together? *Ans.* $\frac{abc}{ab + ac + bc}$ days.

This example is but the generalization of the last four. When we thus have a general solution, to obtain the result in any case, it is only necessary to substitute for the general terms in the result, the corresponding particular numbers, and the answer will be obtained. Thus,

Ex. 39. If John alone can drink a cask of wine in 50 days, Thomas alone in 60 days, and William in 40 days, how long will it last them all?

Here, $a = 50$, $b = 60$, and $c = 40$,

$\therefore$ $abc = 120000$, and $ab + ac + bc = 7400$.

and $$\frac{abc}{ab + ac + bc} = \frac{120000}{7400} = 16\tfrac{8}{37} \text{ Ans.}$$

Ex. 40. A. and B. can perform a piece of work in 12 days. Now, as A. alone could do it in 18 days, in what time could B. do it? *Ans.* 36 days.

Ex. 41. A. and B. working together, can do a piece of work in c days; B. being taken sick before commencing, A. performed it alone in a days. In what time could B. have done it? *Ans.* $\frac{ac}{a - c}$ days.

Ex. 42. A man and his wife usually drank a cask of beer in 12 days, but when the man was absent, his wife emptied it in 30 days. In what time would he alone have finished it? *Ans.* 20 days.

Ex. 43. A courier who travels 60 miles a day had been despatched 5 days, when another who can travel 75 miles a

day, was sent to recall him. How far had the latter to travel before he overtook the former? *Ans.* 1500 miles.

Ex. 44. The tail of a certain fish weighs 9 pounds, the head weighs as much as the tail and one-third of the body, and the body weighs twice as much as the head and tail. What is the weight of the fish? *Ans.* 162 pounds.

Ex. 45. A fish was caught, of which the head weighed 12 lbs., the tail weighed as much as one-half the body diminished by the weight of the head, and the whole fish weighed seven times as much as the tail. What was its weight? *Ans.* 63 lbs.

Ex. 46. A privateer sailing 10 miles an hour pursues a merchantman 18 miles off, which can sail only $8\frac{1}{2}$ miles per hour. How far will the former go before the latter is overtaken? *Ans.* 120 miles.

Ex. 47. A company settling their reckoning, found that each had to pay \$1.75; but before they settled, 4 of them went away, in consequence of which the rest had to pay \$2 apiece. How many were there at first? *Ans.* 32.

Ex. 48. A man in play lost one-fourth of his money, and then won 3 shillings; he afterward lost one-third of what he then had, and gained 5 shillings, when he found he was possessed of 17 shillings. With what sum did he begin? *Ans.* 20s.

Ex. 49. A man goes to a tavern and loses 24 dollars in play; he then borrows as much as he has left, and going to another tavern, loses 24 dollars, and borrows as much as he has left; at a third tavern he spends 24 dollars, when he finds he has only 12 dollars remaining. How much had he at first? *Ans.* 45 dollars.

Ex. 50. In a mixture of wheat and barley, one-third of the whole plus 25 bushels was wheat, and one-half of the whole less 5 bushels was barley. How many bushels were there of each? *Ans.* 65 of wheat, and 55 of barley.

Ex. 51. A. and B. set out to pay some debts, A.'s money being two-thirds of B.'s; after A. had paid 1 dollar less than two-thirds of his money, and B. 1 dollar more than seven-eighths of his, B. had only half as much as A. had left. What sum had each at first?

Ans. A. had \$72, and B. had \$108.

Ex. 52. A shepherd, in time of war, was plundered by a party of the enemy, of one-fourth of his flock and one-fourth of a sheep. This party being driven back, a detachment of the other army took one-third of what he had left and one-third of a sheep more. Fortune having again changed, he was plundered of half his remaining flock and half a sheep, when he had but 25 left. How many had he at first? *Ans.* 103.

Ex. 53. Several detachments of artillery divided a number of cannon balls: the first took 72 and one-ninth of the remainder; the next 144, and one-ninth of the remainder; the third 216, and one ninth of the remainder, and so on, when it was found the balls were equally divided. Determine the number of detachments and balls.

Ans. 4608 balls and 8 detachments.

Ex. 54. Divide 198 into 5 such parts that the first being increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by five, the results shall all be equal.

Ans. 23, 22, 27, 6, and 120.

Ex. 55. If a steamboat can go 28 miles with the current, in one hour and a half, while it requires two hours and a quarter to return, though it keeps near the shore, where the stream is only $\frac{3}{5}$ as rapid as in the middle, it is required to find the velocity of the current in the middle.

Ans. $3\frac{8}{9}$ miles per hour.

Ex. 56. A tenant agreed to pay his landlord $\frac{2}{3}$ of the produce of his farm after deducting necessary expenses, and found that his own share was $\frac{1}{6}$ of the whole. The next year, the price of produce having fallen $\frac{2}{5}$, and the expense of cultivation $\frac{1}{3}$, he found he had to pay his landlord \$400. What was the original value of the produce. *Ans.* \$2250.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

§ **60.** All the equations in the preceding articles have contained but one unknown, and the questions which have been proposed have required the assumption of but one indeterminate quantity for their solution. In many cases, however, the solution is much facilitated by assuming more than one unknown quantity. Whenever this is done, the number of independent equations must be equal to that of the unknown quantities. If there are fewer, the values of the unknown quantities will be indeterminate. If there are more equations, they will be inconsistent with one another.

Thus if we have the equation

$$3x + 6y = 45, \quad x = 15 - 2y,$$

in which result y may have any number of values, which will of course give as many for x.

If to the above equation we add the following,

$$4x - 2y = 30,$$

we shall have $$x = \frac{15 + y}{2}.$$

Now as this value of x must be identical with the preceding, we must have

$$\frac{15 + y}{2} = 15 - 2y,$$

whence $$y = 3,$$

and $$x = 15 - 2y = 15 - 6 = 9,$$

or, $$x = \frac{15 + y}{2} = \frac{15 + 3}{2} = 9.$$

These two values, viz.

$$x = 9, \text{ and } y = 3,$$

will satisfy the original equations,

$$3x + 6y = 45, \text{ and } 4x - 2y = 30,$$

but no others except such as may be obtained from them.

Thus the values

$$x = 9,\ y = 3,$$

will satisfy the equation

$$7x - 4y = 51,$$

which *appears* to be independent of the former ones. It may, however, be obtained from them by subtracting the first from 27 times the second, and dividing the result by 15.

Thus, $3x + 6y = 45$,

27 times the second is $108x - 54y = 810$,

whence, by subtraction, $105x - 60y = 765$,

dividing by 15, $7x - 4y = 51$,

whence it appears this equation is dependent on the others.

ELIMINATION.

§ **61.** When two equations are given, by means of which we are to determine two unknown quantities, the method of proceeding is to exterminate one of the unknown quantities, and thus obtain a single equation involving only one unknown.

The process of exterminating one of the unknowns is called Elimination. It is generally effected by one of three modes.

RULE I.

Make the coefficients of that unknown which it is desired to eliminate, the same in both equations, by multiplying or dividing one or both by such numbers as are necessary for that purpose.

Then take the sum or difference of the resulting equations according as the similar terms have different or like signs, and there will result an equation involving but one unknown. The value of this may be determined as in the preceding examples.

This value substituted in one of the original equations will give an equation that will enable us to determine the other unknown.

EXAMPLES.

Ex. 1. Given $5x - 3y = 18,$
and $7x + 2y = 50,$
to eliminate y, and thence determine the values of x and y.

Multiply the first by 2, and the second by 3, and they become $10x - 6y = 36,$
and $21x + 6y = 150.$

Adding, we obtain $31x = 186,$
whence $x = 6.$

This substituted in the second equation gives
$$42 + 2y = 50,$$
whence $y = 4.$

Ex. 2. Given $3x - \frac{1}{2}y = 15,$
and $\frac{1}{4}x - 3y = -12,$
to determine the values of x and y.

Multiplying the first by 2, and the second by 24, they become $6x - y = 30,$
and $6x - 72y = -288.$

Subtracting $71y = 318,$
and $y = 4\frac{34}{71}.$

This substituted in the first gives
$$3x - 2\frac{17}{71} = 15,$$
whence $x = 5\frac{53}{71}.$

Ex. 3. Given $7x + 5y = 41,$
and $5x + 2y = 23,$
to find the values of x and y. *Ans.* $x = 3, y = 4.$

Ex. 4. Given $6x - 7y = 5,$
and $3x + 2y = 41,$
to find the values of x and y. *Ans.* $x = 9, y = 7$

Ex. 5. Given $11x + 3y = 100,$
and $4x - 7y = 4,$
to find the values of x and y. *Ans.* $x = 8, y = 4.$

Ex. 6. Given $$\frac{x}{2}+\frac{y}{3}=7,$$
and $$\frac{x}{3}+\frac{y}{2}=8,$$
to find the values of x and y. *Ans.* $x=6$, $y=12$.

Ex. 7. Given $$\frac{x}{7}+7y=99,$$
and $$\frac{y}{7}+7x=51,$$
to find the values of x and y. *Ans.* $x=7$, $y=14$.

Ex. 8. Given $$x+15y=53,$$
and $$y+3x=27,$$
to find the values of x and y. *Ans.* $x=8$, $y=3$.

Ex. 9. Given $$9x+\frac{8y}{5}=70,$$
and $$7y-\frac{13x}{3}=44,$$
to find the values of x and y. *Ans.* $x=6$, $y=10$.

RULE 2.

Find the value of one of the unknowns from one equation, and substitute this value for that unknown in the other equation.

EXAMPLES.

Ex. 1. Given $$5x-3y=18,$$
and $$7x+2y=50,$$
to find the values of x and y.

Here, from the first we obtain, by transposition and division,
$$y=\frac{5x-18}{3}.$$

This value substituted in the other equation gives
$$7x+\frac{10x-36}{3}=50,$$

Clearing of fractions and transposing,

$$31x = 186,$$

whence $$x = 6,$$

and $$y = \frac{5x - 18}{3} = \frac{30 - 18}{3} = 4.$$

Ex. 2. Given $$3x - \frac{1}{2}y = 15,$$

and $$\frac{1}{4}x - 3y = -12,$$

to find the values of x and y.

Clearing the first of fractions, and transposing, we obtain

$$y = 6x - 30.$$

Substituting in the second equation, we find

$$\frac{1}{4}x - 18x + 90 = -12.$$

Clearing and transposing,

$$-71x = -408.$$

whence $$x = 5\tfrac{53}{71},$$

and $$y = 6x - 30 = 4\tfrac{34}{71}.$$

Ex. 3. Given $6x - 14y = -6$,

and $5x - 9y = 11$,

to find the values of x and y. *Ans.* $x = 13$, $y = 6$.

Ex. 4. Given $$\frac{2x - y}{2} + 14 = 18,$$

and $$\frac{2y + x}{3} + 16 = 19,$$

to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

Ex. 5. Given $$3x + \frac{7y}{2} = 22,$$

and $$11y - \frac{2x}{5} = 20,$$

to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

Ex. 6. Given $$\frac{x}{6}+\frac{y}{4}=12,$$

and $$\frac{x}{4}+\frac{y}{6}=11\tfrac{1}{3},$$

to find the values of x and y. *Ans.* $x=24$, $y=32$.

Ex. 7. Given $$\frac{3x-1}{5}+3y-4=15,$$

and $$4x-3y=13,$$

to find the values of x and y. *Ans.* $x=7$, $y=5$.

Ex. 8. Given $$4x+\frac{15-x}{4}=2y+5+\frac{7x+11}{16},$$

and $$3y-\frac{2x+y}{5}=2x+\frac{2y+4}{3},$$

to find the values of x and y. *Ans.* $x=3$, $y=4$.

Rule 3.

Find the value of the same unknown in each equation. These values being connected by the sign =, will form an equation from which the value of the other unknown may be found.

Examples.

Ex. 1. Given $$5x-3y=18,$$
and $$7x+2y=50,$$
to find the values of x and y.

From the first, $$y=\frac{5x-18}{3},$$

from the second, $$y=\frac{50-7x}{2}.$$

Equating these values, we have
$$\frac{5x-18}{3}=\frac{50-7x}{2}.$$

Clearing of fractions, $10x-36=150-21x$.
whence, $$31x=186,$$
and $$x=6,$$
$$\therefore\quad y=\frac{5x-18}{3}=4.$$

Ex. 2. Given $4x - 7y = 7$,
and $6x + 2y = 48$,
to find the values of x and y. *Ans.* $x = 7, y = 3$.

Ex. 3. Given $4x - 15 = \frac{11}{3}y$,

and $\frac{2x - y}{3} + 16 = 4x - 3y$,

to find the values of x and y. *Ans.* $x = 12, y = 9$.

Ex. 4. Given $\frac{7 + x}{5} - \frac{2x - y}{4} = 3y - 5$,

and $\frac{5y - 7}{2} + \frac{4x - 3}{6} = 18 - 5x$

to find the values of x and y. *Ans.* $x = 3, y = 2$.

Ex. 5. Given $\frac{4}{5 + y} = \frac{5}{12 + x}$,
and $4x + 10y = 7$,
to find the values of x and y. *Ans.* $x = -3\frac{1}{4}, y = 2$.

EXAMPLES PRODUCING EQUATIONS OF THE FIRST DEGREE INVOLVING TWO UNKNOWN QUANTITIES.

§ **62.** Having explained, in the preceding article, the three principal methods of elimination, we shall annex some examples to be solved by the use of two unknowns. Some of these may be solved with the use of but one letter; in fact, by a proper arrangement of the preliminary steps, all may be so solved; but generally, the attempt is attended with considerable difficulties, which are frequently entirely removed by making another assumption.

Ex. 1. What fraction is that, to the numerator of which if 1 be added, the value shall be $\frac{1}{3}$; but if 1 be added to the denominator, the value shall be $\frac{1}{4}$.

Here let $\frac{x}{y}$ represent the fraction.

Then, by the conditions of the problem,

$$\frac{x + 1}{y} = \frac{1}{3}, \text{ and } \frac{x}{y + 1} = \frac{1}{4}.$$

Clearing of fractions and transposing, we have

$$3x - y = -3,$$

and $$4x - y = 1.$$

Subtracting $$x = 4,$$

whence $$y = 3x + 3 = 15.$$

The fraction is therefore $\frac{x}{y} = \frac{4}{15}$

Ex. 2. At a certain election, 2537 votes were polled, the successful candidate having a majority of 147. What number did each receive?

Here, if x and y represent the numbers received by the successful and unsuccessful candidates respectively, we shall have $$x + y = 2537,$$

and $$x - y = 147,$$

whence, by addition and subtraction, we obtain

$$2x = 2684,$$

and $$2y = 2390,$$

$$\therefore \quad x = 1342, \text{ and } y = 1195.$$

Ex. 3. Two purses contain together \$750. If we take \$125 from one, and put it into the other, the second will contain twice as much as the first. What did each contain?

Here, if x and y represent the sums in each purse, we shall have $$x + y = 750,$$

and $$y + 125 = 2(x - 125) = 2x - 250.$$

From the last equation, by transposition, we obtain

$$2x - y = 375,$$

by adding this to the first equation, we have

$$3x = 1125,$$

and $$x = 375,$$

consequently, $$y = 750 - x = 375.$$

Ex. 4. What two numbers are there, whose sum is 75, and difference 33? *Ans.* 54 and 21.

Ex. 5. A man bought 4 bushels of wheat and 7 bushels of rye for \$9.45. He afterwards bought 5 bushels of wheat and 6 of rye at the same rate, for \$9.75. What was the price per bushel of each?

Ans. Wheat \$1.05, and rye 75 cts.

Ex. 6. A farmer has a bin which will hold 100 bushels. This he wishes to fill with a mixture of corn and oats. Now if he puts in 40 bushels of oats, the whole will be worth 52 cents per bushel; but if he puts in 40 bushels of corn, the mixture would be worth 48 cents per bushel. What are the corn and oats worth respectively?

Ans. Corn 60 cts., and oats 40 cts. per bushel.

Ex. 7. A person one day hired 5 men and 7 boys to work for him, paying them \$8.75. The next day he hired at the same rate 9 men and 4 boys, and paid them \$10.37½. What are the wages of each?

Ans. Men 87½ cts., boys 62½ cts.

Ex. 8. One-third of the number of sheep, added to one-fourth of the number of cows in a certain meadow, makes 16. And one-fourth of the number of sheep and one-eighth of the number of cows makes 11. How many were there of each? *Ans.* 36 sheep and 16 cows.

Ex. 9. A person purchased a certain number of yards of flannel at 90 cts. per yd., and a certain number of yards of linen at 60 cts. per yd., paying for the whole, \$38.40. She found afterwards that she could have bought as much flannel as she had of linen, and as much linen as she had of flannel, for \$35.10. What quantity of each did she buy?

Ans. 30 yds. of flannel, and 19 of linen.

Ex. 10. A vintner sold to one man 20 dozen port wine and 30 dozen sherry, for \$360, and to another, 30 dozen port and 20 dozen sherry, for \$390. What was the price per dozen of each sort? *Ans.* Port \$9, and sherry \$6.

Ex. 11. A person has two horses, and a carriage worth \$200. Now, if he puts the first horse to the carriage, the

whole is worth twice as much as the second horse. But if he puts the second horse to the carriage, the two are worth three times as much as the first. What is the value of each horse? *Ans.* One \$120, and the other \$160.

Ex. 12. I bought at one time 7 yards of linen and 16 yards of muslin, for \$6, and at another time 15 yards of linen and 6 yards of muslin at the same rate for \$9.18. What was the price per yard of each?

Ans. Linen 56 cts. and muslin 13 cts.

Ex. 13. A farmer purchased 15 steers and 9 cows, for \$732. At another time he bought 19 steers and 6 cows for \$842, paying \$3 a head more for the steers, and \$3 a head less for the cows than before. What was the first price of each? *Ans.* Steers \$35, and cows \$23 per head.

Ex. 14. After A. had won \$4 of B., he had only half as much as B. had left; but if B. had won \$6 of A., he would have had three times as many as A. had left. What sum had each? *Ans.* A. had \$36, and B. \$84.

Ex. 15. A merchant bought 40 yards of linen and 70 yards of muslin for \$25. He afterwards sold 25 yards of the linen and 37 yards of the muslin for \$18.12½, making 9 cents a yard on the former and 2½ cents on the latter. What did he give per yard for each?

Ans. Linen 45 cts., and muslin 10 cts. per yard.

Ex. 16. John has in his purse 35 coins, consisting of 12½ and 3 cent pieces, the whole value being \$2.66½. How many has he of each?

Ans. 17 of the former, and 18 of the latter.

Ex. 17. A certain sum of money was to be divided among the boys in a class. Now, had there been three more, each boy would have received 5 cents less, and if there had been two fewer, each would have received 5 cents more than he did. Required the number of boys, and the sum received by each

Ans. 12 boys, and each received 25 cents.

Ex. 18. Two men start on a journey; at the end of the first day, A. is 12 miles ahead of B.; at the end of 9 days, (A. having laid by two days,) B. is 12 miles in advance. At what rate did they travel?

Ans. A. 60 miles per day, and B. 48 miles per day.

Ex. 19. A person has some guineas and some moidores in his purse, the whole amount being £93; but a servant having robbed him of one-sixth of his moidores, and three-fifths of his guineas, left him only £54 15s.. How many of each had he at first, a moidore being worth 27s. and a guinea 21s.? *Ans.* 50 guineas, and 30 moidores.

Ex. 20. A reservoir is supplied by two pumps, which together will fill it in 18 hours. After they had been in operation 5 hours, one of them gave out, and the other filled the reservoir in 21h. 40m. In what time would either alone have filled it? *Ans.* 45 hours, and 30 hours.

Ex. 21. A man and his wife usually drank a cask of beer in 15 days, but after drinking 6 days, the woman emptied it in 30 days. In what time would either alone have emptied it?

Ans. The man in $21\frac{3}{7}$ days, and the woman in 50 days.

Ex. 22. Some smugglers discovered a cave which would exactly hold their cargo, consisting of 13 bales of cotton and 33 casks of rum. Before they had finished unloading, a revenue cutter coming in sight, they sailed away with 9 casks and 5 bales, having filled the cave two-thirds full. How many bales, or how many casks would it hold?

Ans. 24 bales, or 72 casks.

Ex. 23. A vintner wishes to fill a barrel containing 32 gallons with wine which shall cost him $1.25 per gallon. He has some worth 85 cents, and some of another quality, worth $1.40. How many gallons of each must he put in the barrel?

Ans. $23\frac{3}{11}$ galls. at $1.40, and $8\frac{8}{11}$ galls. at 85 cents.

Ex. 24. A miller has two qualities of feed, worth respec-

tively, 56 and 35 cents per bushel, with which he wishes to fill a bin that will hold 90 bushels, so that the mixture may be worth 45 cents per bushel. How much of each must he put in?

Ans. $42\frac{6}{7}$ bushels of the first, and $47\frac{1}{7}$ of the second.

Ex. 25. Find two numbers such that one-third the first and one-fourth the second shall be 11, and one-fifth the first and one-eighth the second shall be 6. *Ans.* 15 and 24.

Ex. 26. There are two numbers such that one-half the less diminished by one-eighth the greater shall be 8, but three-sevenths of the greater diminished by three-fifths the less shall be 6. What are the numbers? *Ans.* 56 and 30.

Ex. 27. There is a certain fraction, such that if the numerator be increased by 2 the value becomes one-third, but if the denominator be diminished by 3 the value becomes one-fourth. What is the fraction? *Ans.* $\frac{3}{15}$.

Ex. 28. A rectangular bowling-green being measured, it was found that if it were 5 rods broader and 4 rods longer it would contain 116 perches more; but if it were 4 rods broader and 5 rods longer, it would contain 113 perches more. Required the length and breadth. *Ans.* Length 12, and breadth 9 r.

Ex. 29. A. and B. have each a flock of sheep; at the end of the year, A.'s having been increased by 80, and B.'s diminished by 20, B.'s flock is only $\frac{3}{8}$ of A.'s. Now had A lost 20 of his, and B increased his by by 90, B.'s would have been $\frac{10}{7}$ of A.'s. What was the number in each flock?

Ans. A.'s 160, and B.'s 110.

SIMPLE EQUATIONS CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

§ **63.** There are frequently found problems which require for their easy solution more than two unknowns. The solution of equations which contain three or more unknowns, is performed on precisely the same principles as where there are but two, the first method being that most generally employed.

As an example, we may solve the following equations:

$$3x - 4y + 5z = 14,$$
$$6x + 2y - 7z = -10,$$
$$x + 3y - 4z = -5.$$

To eliminate x from the first two equations, we subtract the second from twice the first; there remains

$$-10y + 17z = 38.$$

In like manner, by subtracting three times the third from the first, we obtain the equation

$$-13y + 17z = 29.$$

z may be eliminated from these two equations by subtracting the latter from the former. By this means, we find

$$3y = 9.$$

whence $y = 3$,

consequently, $-30 + 17z = 38$,

whence $17z = 68$,

and $z = 4$.

The values of y and z thus obtained being substituted in either of the original equations, will enable us to find x. As the third equation is the simplest, it had better be employed; there results

$$x + 9 - 16 = -5,$$

whence $x = 2$.

From this solution we obtain the following

RULE.

1. *Selecting one of the unknown quantities; eliminate it from different pairs of the given equations, by one of the rules,* § 61, *until a series of equations equal in number to that of the remaining unknown quantities is obtained.*

2. *Eliminate another unknown quantity by combining in like manner these last equations.*

3. *Continue this operation until there remains an equation with but one unknown quantity, the value of which may readily be found. Substitute this value in one of the equations which contain but two unknowns, and a second value is obtained; so proceeding backward, the values of all the unknowns will be obtained in succession.*

§ 64. Should any of the equations not contain all the unknown quantities, it will generally be best to eliminate those unknowns that are found in but part of the equations, and combine the resulting equations with the remaining original equations.

In all cases there must be obtained as many independent equations as there are unknown quantities to be determined.

Examples.

Ex. 1. Given
$$5x + 4y - 7z = 30,$$
$$3x + 7y = 36,$$
and
$$4y - 5z = 7,$$
to find the values of x, y and z.

Here, as x is not contained in the third equation, we shall eliminate it from the first and second, by subtracting three times the first from five times the second; there results
$$23y + 21z = 90.$$

Eliminating y from this and
$$4y - 5z = 7,$$
we obtain
$$199z = 199,$$
whence
$$z = 1.$$

This substituted in $4y - 5z = 7$,
gives
$$4y = 12,$$
$$y = 3,$$
and consequently
$$3x = 36 - 7y = 15,$$
$$x = 5.$$

Ex. 2. Given
$$x + y + z = 31,$$
$$x + y - z = 25,$$
and
$$x - y - z = 9,$$
to find the values of x, y and z.

Ans. $x = 20, y = 8$, and $z = 3$

Ex. 3. Given
$$x + y + z = 29,$$
$$x + 2y + 3z = 62,$$
and
$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10,$$
to find the values of x, y and z.

Ans. $x = 8, y = 9$, and $z = 12$

Ex. 4. Given $x + y = 10,$
$x + z = 19,$
and $y + z = 23,$
to find the values of x, y and z.

Ans. $x = 3, y = 7$, and $z = 16$.

Ex. 5. Given $x + y + z = 90,$
$2x + 40 = 3y + 20,$
and $x + 4z = 135,$
to find the values of x, y and z.

Ans. $x = 35, y = 30$, and $z = 25$

Ex. 6. Given $$\frac{1}{x} + \frac{1}{y} = a,$$
$$\frac{1}{x} + \frac{1}{z} = b,$$
and $$\frac{1}{y} + \frac{1}{z} = c,$$
to find the values of x, y and z.

Ans. $x = \dfrac{2}{a + b - c}, y = \dfrac{2}{a - b + c}$, and $z = \dfrac{2}{b + c - a}$.

Ex. 7. Given $x + y + z + u = 2,$
$8x + 4y + 2z + u = 5\frac{1}{2},$
$27x + 9y + 3z + u = 13,$
and $64x + 16y + 4z + u = 26,$
to find the values of x, y, z and u.

Ans. $x = \frac{1}{4}, y = \frac{1}{2}, z = \frac{1}{4}$, and $u = 1$

Ex. 8. Given $$\frac{2x + 3y}{x + y} = 2\tfrac{1}{5},$$
$$\frac{x + z}{5x - 5z} = \tfrac{1}{3},$$
and $$\frac{10x - 3z}{16y - 2z} = \frac{2x + 5}{14},$$
to find the values of x, y and z.

Ans. $x = 16, y = 4$, and $z = 4$.

§ **65.** EXAMPLES INVOLVING MORE THAN TWO UNKNOWNS.

Ex. 1. There are four casks of equal capacity; in the first have been put 7 gallons of brandy, 8 gallons of wine, and 16 gallons of water; in the second there are 14 gallons of brandy, 10 gallons of wine, and 7 gallons of water; and in the third are 11 gallons of brandy, 13 gallons of wine, and 7 gallons of water. How much must be taken from each of these, to fill the fourth so that it shall contain 9 gallons of brandy, 9 gallons of wine, and 13 gallons of water?

Let x, y and z represent the number of gallons necessary to be taken from the three casks respectively.

Then the x gallons will contain

$\frac{7x}{31}$ gallons of brandy, $\frac{8x}{31}$ gallons of wine,

and $\frac{16x}{31}$ gallons of water.

Similarly, the y gallons contain

$\frac{14y}{31}$ gallons of brandy, $\frac{10y}{31}$ gallons of wine,

and $\frac{7y}{31}$ gallons of water.

and the z gallons contain

$\frac{11z}{31}$ gallons of brandy, $\frac{13z}{31}$ gallons of wine,

and $\frac{7z}{31}$ gallons of water.

Consequently,

$$\frac{7x}{31} + \frac{14y}{31} + \frac{11z}{31} = 9,$$

$$\frac{8x}{31} + \frac{10y}{31} + \frac{13z}{31} = 9,$$

and
$$\frac{16x}{31} + \frac{7y}{31} + \frac{7z}{31} = 13,$$

or, clearing of fractions, we have

$$7x + 14y + 11z = 279,$$

$$8x + 10y + 13z = 279,$$

and
$$16x + 7y + 7z = 403.$$

Subtracting seven times the second from eight times the first, we obtain

$$42y - 3z = 279.$$

Likewise, by subtracting the third equation from twice the second, there results

$$13y + 19z = 155.$$

Multiplying these respectively by 19 and 3, we have

$$798y - 57z = 5301,$$

and $$39y + 57z = 465,$$

whence, by addition, $$837y = 5766,$$

and $$y = 6\tfrac{8}{9}$$

This value substituted in the equation

$$13y + 19z = 155,$$

gives $$\frac{806}{9} + 19z = 155,$$

whence $$171z = 589,$$

and $$z = 3\tfrac{4}{9}.$$

These values of y and z substituted in the first equation give

$$7x + \frac{868}{9} + \frac{341}{9} = 279,$$

whence $$63x = 1302,$$

and $$x = 20\tfrac{2}{3}.$$

There must therefore be $20\tfrac{2}{3}$ gallons taken from the first $6\tfrac{8}{9}$ gallons from the second, and $3\tfrac{4}{9}$ gallons from the third.

Ex. 2. There are three numbers, such that half the first, one-third the second, and one-fourth the third, shall make 18; one-fourth the first, one-fifth the second, and one-sixth the third, shall make 10; and one-fifth the first, the second, and one-third the third, shall make 23. Required the numbers. *Ans.* 20, 15, and 12.

Ex. 3. An estate is to be divided amongst three children, so that the shares of A. and B. together shall be equal to C.'s, increased by \$1000; the shares of A. and C. shall be \$3000 more than three times B.'s, and the shares of B. and C. shall be twice A.'s. Required the shares.

Ans. A.'s, \$5000; B.'s, 3000; and C.'s, \$7000

Ex. 4. There are three numbers, such that one-half the first, one-third the second, and one-fourth the third make 18; one-third the first, one-sixth the second, and one-eighth the third make 10; and one-sixth the first, one-ninth the second, and one-tenth the third make $6\frac{2}{5}$. What are they? *Ans.* 12, 18 and 24.

Ex. 5. A. and B. can do a piece of work in 8 days, A. and C. in 9 days, and B. and C. in 10 days. What time would it take each to perform it alone?

Ans. A. $14\frac{34}{49}$, B. $17\frac{23}{41}$, C. $23\frac{7}{31}$.

Ex. 6. Required to find three numbers, such that half the first added to the sum of the others may be 25; one-third the second added to the sum of the others may be 23, and one-fourth the third added to the sum of the others may be 20. *Ans.* 8, 9, and 12.

Ex. 7. Three farmers conversing about their cattle, says A. to B., if I were to give you 5 of my oxen, and then give C. $\frac{3}{17}$ of what I had left, and you were to give him $\frac{1}{15}$ of the number you then had, we all three would have the same number. Now as they altogether had 210, it is required to determine the number owned by each.

Ans. A. had 90, B. 70, and C. 50.

CHAPTER VI.

QUADRATIC EQUATIONS, OR EQUATIONS OF THE SECOND DEGREE.

§ **66.** A quadratic, or equation of the second degree, is one which involves the square of the unknown. Thus,

$$7x^2 = 56, \text{ and } 4x^2 - 3x + 12 = 64,$$

are quadratic equations.

§ **67.** A quadratic which contains only the square of the unknown joined with given quantities, is called a *pure equation;* one which contains the first and second powers, is an *adfected quadratic.*

PURE EQUATIONS.

68. The solution of *pure equations*, whether quadratic or of higher degree, is effected by separating the *square* or complete power, and then extracting the root. Thus, if the equation

$$7x^2 - 16 = 5x^2 + 82$$

were given, it would be necessary first to transpose the terms; we would thus obtain

$$2x^2 = 98,$$

whence $\qquad x^2 = 49,$

and extracting the square root,

$$x = \pm 7.$$

§ **69.** In the result of the last article we have the value

$$x = \pm 7,$$

thus giving to x two values numerically equal, but of contrary signs. This result arises from the general principle, that the even roots always have the double signs; in other words, that they may be either positive or negative. In the case above,

$$7 \times 7 = 49, \text{ and } -7 \times -7 = 49,$$

the square root of 49 may be either $+7$, or -7.

§ **70.** From what has been said above, it is easy to see the reason of the following

Rule for solving pure equations.

1st. *Transpose the terms so that the power of the unknown shall stand by itself, forming the left-hand member of the equation.*

2d. *Divide by the coefficients of the power, and then extract the proper root of the resulting equation.*

Examples.

Ex. 1. Given $\quad 8x^2 - 57 + 3x^2 = 4x^2 + 6,$

to find the value of x.

First, by transposition, we have

$$8x^2 + 3x^2 - 4x^2 = 6 + 57,$$

whence $\qquad 7x^2 = 63$, and $x^2 = 9$,

whence, extracting the square root,

$$x = \pm 3.$$

Ex. 2. Given $5y^3 + 17 - 3y^3 = 12y^3 - 623$, to find the value of y.

By transposition, $-10y^3 = -640$, whence dividing by -10, we have

$$y^3 = 64 \text{ and } y = 4.$$

Ex. 3. What are the values of x which satisfy the equation

$$6 + 4x^2 = \frac{8x^2}{7} + 5x^2 - 64?$$

Ans. $x = 7$, or -7.

Ex. 4. Given $\frac{4}{x} + \frac{x}{4} = \frac{x}{2}$, to find the values of x. *Ans.* $x = 4$, or $x = -4$.

Ex. 5. Given $\frac{x-4}{15} + \frac{30}{x-4} = \frac{19(x-4)}{15}$, to find the values of x.

Here, clearing of fractions, we obtain

$$(x-4)^2 + 450 = 19(x-4)^2,$$

whence $\qquad 18(x-4)^2 = 450,$

$\therefore \qquad (x-4)^2 = 25.$

Extracting the square root, we have

$$x - 4 = \pm 5,$$

and $\qquad x = 4 \pm 5 = 9$, or -1.

Ex. 6. Given $(x+2)^2 = 4x + 5$, to find the values of x. *Ans.* $x = \pm 1$.

Ex. 7. The expenses of a certain company amounted to forty-five dollars, each person having spent one-fifth as many dollars as there were persons in the company. How many persons were there? *Ans.* 15.

Ex. 8. What number is that, which being divided by 7, and the result multiplied by five times the number itself, the product shall be $\frac{45}{112}$. *Ans.* $\pm\frac{3}{4}$.

Ex. 9. It is required to divide 75 into two such parts, that each being divided by the other, the difference of the quo tients shall be equal to 2.

Ans. The parts are 53·033 and 21·967.

Ex. 10. What two numbers are those which are to each other as 5 to 6, and the sum of whose squares is 244?

Ans. 10 and 12.

Ex. 11. Says A. to B., my age is five times yours, and the sum of the squares of our ages is 5850 years. How old were they? *Ans.* A. 75 years, and B. 15 years.

Ex. 12. The length of a rectangular tract of land is two and one-third times its breadth, and the area is 52 A. 2 R. Required the length and breadth.

Ans. Length, 140 rods, and breadth, 60 rods.

Ex. 13. The length of a rectangular tract of land ex ceeds the breadth by 14 rods; and the area is 20 acres Required the length and breadth.

Ans. Length, 64 rods, and breadth, 50.

NOTE.—Assume $x+7$ and $x-7$, for the length and breadth.

§ **71.** If there are two unknown quantities, and they cannot both be included in a single power—

Eliminate one of them as in simple equations, and then proceed as in last article.

EXAMPLES.

Ex. 1. Given $x+y=10$,

and $xy=21$,

to find the values of x and y.

Here, if we square the first, we shall have

$$x^2+2xy+y^2=100,$$

but

$$4xy \quad = 84,$$

therefore, by subtraction, $x^2-2xy+y^2=16$.

As this is a complete square, extract the square root. and we have $x-y=4$

Adding this to, and subtracting it from the equation

$$x + y = 10,$$

we have $2x = 14$, and $2y = 6$,

whence $x = 7$ and $y = 3$.

Ex. 2. Given $x^2 - y^2 = a$,

and $x^2 + y^2 = b$,

to find the values of x and y.

Here, by addition and subtraction, we have

$$2x^2 = b + a,$$

and $2y^2 = b - a$,

whence $x^2 = \frac{b+a}{2}$,

and $y^2 = \frac{b-a}{2}$,

$\therefore \quad x = \sqrt{\left(\frac{b+a}{2}\right)}$, and $y = \sqrt{\left(\frac{b-a}{2}\right)}$.

Ex. 3. What two numbers are those, whose product is 48, and the quotient arising from dividing the greater by the less is 3? *Ans.* 12 and 4.

Ex. 4. There are two numbers, whose sum is 40, and their product 300. What are they? *Ans.* 30 and 10.

Ex. 5. The product of two numbers is 75, and the quotient of the second by the first 3. What are they?

Ans. 5 and 15.

Ex. 6. Given $x^2 + xy = 70$,

and $xy + y^2 = 30$,

to find the values of x and y.

Here, by addition, we obtain

$$x^2 + 2xy + y^2 = 100,$$

whence, extracting the square root, we have

$$x + y = 10.$$

Dividing this into the original equations, gives

$$x = 7,$$

$$y = 3.$$

Ex. 7. Required to find two numbers, such that their difference multiplied by the greater may be 36, and multiplied by the sum may be 63. *Ans.* 12 and 9.

Ex. 8. The sum of the squares of two numbers is 89, and their product 40. What are they? *Ans.* 8 and 5.

Here, if x and y are the numbers, the equations are,

$$x^2 + y^2 = 89,$$

and

$$xy = 40.$$

By adding to, and subtracting from the first twice the second, we obtain

$$x^2 + 2xy + y^2 = 169,$$

and

$$x^2 - 2xy + y^2 = 9,$$

whence extracting the square root,

$$x + y = 13,$$

and

$$x - y = 3.$$

From these we find, by addition and subtraction,

$$2x = 16,$$

and

$$2y = 10,$$

whence,

$$x = 8, \text{ and } y = 5.$$

Ex. 9. What two numbers are those, whose difference multiplied by the greater, and then divided by the less, gives 24, but whose difference, multiplied by the less, and the product divided by the greater, gives 6?

Ans. 24 and 12.

Ex. 10. What two numbers are those, whose product is 180, and their difference 3? *Ans.* 15 and 12.

Ex. 11. A man bought two pieces of muslin, each of which cost as many cents per yard as there were yards in its length. Now the whole number of yards being 36, and the whole price of one piece four times that of the other, required their length. *Ans.* 24 and 12 yards.

Ex. 12. Divide 49 into two such parts that the quotient of the greater, divided by the less, may be $\frac{16}{9}$ that of the less divided by the greater. *Ans.* 28 and 21

Ex. 13. The sum of two numbers is two and a half times the less, and their difference multiplied by the difference of their squares is 135. Required the numbers.

Ans. 6 and 9.

Ex. 14. A rectangular field, whose length is to its breadth as 6 is to 5, being one-sixth occupied by an orchard, there remains for tillage 625 square perches. What are its dimensions? *Ans.* 30 and 25 perches.

Ex. 15. The difference of two numbers is 7, and the difference of their cubes 1603. What are the numbers?

Ans. 12 and 5.

Note.—This question is best solved by putting $x + y$ for the greater, and $x - y$ for the less.

Ex. 16. The difference of the squares of two numbers is 9, and the difference of their fourth powers 369. What are they? *Ans.* 5 and 4.

Ex. 17. At what rate per cent. will \$4000 amount to \$4630.50 in three years, at compound interest?

Ans. 5 per cent.

Ex. 18. A detachment of soldiers being required for a particular purpose, each company furnished four times as many men as there were companies in the regiment; but these being found insufficient, each company furnished three more men, by which means the number in the detachment was increased by one-sixteenth. How many companies were there? *Ans.* 12.

Ex. 19. The difference of the cubes of two numbers is 56, and the difference of the numbers is equal to 16 divided by their product? What are they? *Ans.* 4 and 2.

Ex. 20. There are two numbers, such that the product of the greater and cube of the less is equal to four-ninths of the product of the less by the cube of the greater, and the sum of the cubes is 35. What are they?

Ans. 3 and 2.

Ex. 21. A man has two ditches to dig, one of them being 50 yards longer than the other. He is to receive for each as many cents per yard as there are yards in its length. How long were they, the whole sum received being $625.

Ans. 200 and 150 yards.

ADFECTED QUADRATIC EQUATIONS.

§ **72.** It has been said (§ 67) that an *adfected quadratic* contains the square and the simple power of the unknown, besides a given term. Every such equation may be reduced to one of three terms, of the form

$$ax^2 + bx = c.$$

§ **73.** The solution of every such equation depends on the principle of making the left-hand member a square, by the addition of a proper quantity. This process is called *completing the square.*

§ **74.** For convenience of solution, equations of the kind under consideration may be divided into several classes, as in the following table, in which the several forms are given in the order in which they will be treated of in the following pages. It must not be understood that there is any essential difference in these forms. In fact any one may, by a simple transformation, be reduced to one of the other forms; consequently, any of the rules of solution will apply to all the cases.

Forms of Quadratics.

1st form,	$x^2 + 2nx = b,$
2d "	$mx^2 + 2nx = b,$
3d "	$mx^2 + nx = b,$

§ **75.** SOLUTION OF EQUATIONS WHICH MAY BE REDUCED TO THE FORM

$$x^2 \pm 2nx = b.$$

We readily find, by squaring, that

$$(x + n)^2 = x^2 + 2nx + n^2.$$

In the complete square of this form, therefore, the third term is the square of half the coefficient of x. Hence we have the following

Rule for completing the square.

Add the square of half the coefficient of x to each member of the equation.

When the square has thus been completed, the solution may be effected by extracting the square root, and then transposing the terms, as in simple equations.

Examples.

Given $x^2 - 8x = 48.$

Here, the coefficient of x is 8; adding, therefore,

$$4^2 = 16$$

to each member, the equation becomes

$$x^2 - 8x + 16 = 64,$$

whence, by extracting the square root,

$$x - 4 = \pm 8,$$

consequently, $x = 4 \pm 8 = 12$, or -4.

In this example we see there are two values of x, or *roots of the equation*, viz. 12 and -4.

That both of these satisfy the conditions may be readily seen by substitution. Thus if $x = 12$, we have

$$12^2 - 8 \times 12 = 144 - 96 = 48.$$

In like manner, by the substitution of -4 for x, the equation becomes

$$(-4)^2 - (8 \times -4) = 16 + 32 = 48.$$

Either value therefore satisfies the equation.

§ **76.** By the solution of the general equation

$$x^2 + 2nx = b,$$

it will be seen that every quadratic must have two values For by completing the square, the equation becomes

$$x^2 + 2nx + n^2 = b + n^2,$$

consequently, extracting the root, we have

$$x + n = \sqrt{b + n^2}.$$

Now since every square root may be taken either positive or negative, this result may be either

$$x + n = \sqrt{b + n^2},$$

or $$x + n = -\sqrt{b + n^2},$$

whence $$x = -n + \sqrt{b + n^2},$$

and $$x = -n - \sqrt{b + n^2}.$$

EXAMPLES.

§ 77. Ex. 1. Given $x^2 - 16x + 91 = 28$.
First, by transposition, we have

$$x^2 - 16x = -63,$$

whence, by completing the square,

$$x^2 - 16x + 64 = 1.$$

Extracting the square root,

$$x - 8 = \pm 1,$$

Consequently, $x = 8 \pm 1 = 9$, or 7.

Ex. 2. Given $7x^2 + 56x - 64 = 27$.
Transposing and dividing by 7, the equation becomes

$$x^2 + 8x = 13,$$

whence, completing the square,

$$x^2 + 8x + 16 = 29.$$

Extracting the square root,

$$x + 4 = \pm\sqrt{29},$$

Consequently, $x = -4 + \sqrt{29}$, or $-4 - \sqrt{29}$.

In this example, the values cannot be expressed accurately in numbers, since $\sqrt{29}$ cannot be obtained. The roots may be approximated to, with any degree of accuracy, by carrying the square root to decimal places.

Ex. 3. Given $x^2 + 16ax = 17a^2$,
to find the values of x.

Here the coefficient is $16a$. Adding, therefore,

$$(8a)^2 = 64a^2$$

to both members of the equation, it becomes

$$x^2 + 16ax + 64a^2 = 81a^2,$$

whence, extracting the square root, $x + 8a = \pm 9a$,

and $x = -8a \pm 9a = a$, or $-17a$.

Ex. 4. Given $x^2 - (4a + 2b)x = -8ab$,
to find the values of x.

The coefficient being $4a + 2b$, we must add

$$(2a + b)^2 = 4a^2 + 4ab + b^2$$

to each member. We thus obtain

$$x^2 - (4a + 2b)x + (2a + b)^2 = 4a^2 - 4ab + b^2,$$

whence, extracting the square root,

$$x - (2a + b) = \pm(2a - b) = \pm 2a \mp b,$$

Consequently, $x = \pm 2a \mp b + 2a + b$
$= 4a$, or $2b$.

Ex. 5. Given $x^2 - 7x = -12$,
to find the values of x. *Ans.* $x = 3$ or 4.

Ex. 6. Given $3x^2 - 40x = 240 + 8x$,
to find the values of x. *Ans.* $x = 20$ or -4.

Ex. 7. Given $\frac{3x^2}{2} + \frac{26x}{3} = 138\frac{5}{6} + \frac{x^2}{3} - 3x$,
to find the values of x. *Ans.* $x = -17$ or 7.

Ex. 8. Given $\frac{x}{4} - \frac{21 + x}{7} = \frac{253}{4x} - \frac{x}{7}$,
to find the values of x. *Ans.* $x = 23$ or -11.

Ex. 9. Given $5x^2 - 4x + 3 = 159$,
to find the values of x.

Transpose 3, and divide by 5, and this equation becomes

$$x^2 - \frac{4}{5}x = \frac{156}{5},$$

whence, by adding

$$\left(\frac{2}{5}\right)^2 = \frac{4}{25},$$

we obtain $x^2 - \frac{4}{5}x + \frac{4}{25} = \frac{156}{5} + \frac{4}{25} = \frac{784}{25}$,

and extracting the square root, $x - \frac{2}{5} = \pm\frac{28}{5}$,

$$\therefore \quad x = \frac{30}{5} = 6, \text{ or } -\frac{26}{5} = -5\tfrac{1}{5}.$$

Ex. 10. Given $3x^2 + 2x - 9 = 76,$

to find the values of x. *Ans.* $x = 5$ or $-\frac{17}{3}$.

Ex. 11. Given $3x - \frac{1121 - 4x}{x} = 2,$

to find the values of x. *Ans.* $x = 19$ or $-\frac{59}{3}$.

Ex. 12. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9},$

to find the values of x. *Ans.* $x = 3$ or $\frac{21}{11}$.

Ex. 13. Given $\frac{3x - 4}{x - 4} + 1 = 10 - \frac{x - 2}{2},$

to find the values of x. *Ans.* $x = 12$ or 6.

§ 78. SOLUTION OF EQUATIONS OF THE FORM

$$mx^2 + 2nx = b.$$

RULE.

Multiply the equation by the coefficient of x^2, and add to the result the square of half the coefficient of x in the original equation. Then proceed as before.

EXAMPLES.

Ex. 1. Given $7x^2 + 16x = 15,$

to find the values of x.

Multiplying by 7, we obtain

$$49x^2 + 112x = 105,$$

add $$8^2 = 64,$$

and the equation becomes

$$49x^2 + 112x + 64 = 169,$$

whence, extracting the square root,

$$7x + 8 = \pm 13,$$

consequently, $7x = -8 \pm 13 = 5$, or -21,

and $$x = \frac{5}{7}, \text{ or } -3.$$

NOTE.—This method is preferable to the former, when the coefficient of the second term is an even number, and not divisible by that of x^2.

Ex. 2. Given $$5x^2 - 8x = 21,$$
to find the values of x. *Ans.* $x = 3$ or $-\frac{7}{5}$.

Ex. 3. Given $$7x^2 + 26x = 33,$$
to find the values of x. *Ans.* $x = 1$ or $-4\frac{5}{7}$.

Ex. 4. Given $$5x^2 - 24x = 5,$$
to find the values of x. *Ans.* $x = 5$ or $-\frac{1}{5}$.

Ex. 5. Given $$\frac{8x}{x+2} - 6 = \frac{20}{3x},$$
to find the values of x. *Ans.* $x = 10$ or $-\frac{2}{3}$.

Ex. 6. Given $$x^2 - \frac{10}{11}x = 30\frac{6}{11},$$
to find the values of x. *Ans.* $x = 6$ or $-5\frac{1}{11}$.

Ex. 7. Given $$\frac{8}{x} - \frac{x}{x^2 - 10} = \frac{8}{x^2 - 10},$$
to find the values of x. *Ans.* $x = 4$ or $-2\frac{6}{7}$.

Ex. 8. Given $$\frac{x+4}{x+6} + \frac{5}{2x+4} = \frac{3x+7}{3x+4},$$
to find the values of x. *Ans.* $x = 8$ or $-\frac{2}{3}$.

Ex. 9. Given $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}$,

to find the values of x. *Ans.* $x=2$ or $\frac{58}{13}$.

Ex. 10. Given $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$,

to find the values of x. *Ans.* $x=6$ or $\frac{40}{13}$.

Ex. 11. Given $\frac{40}{x-5}+\frac{27}{x}=13$,

to find the values of x. *Ans.* $x=9$ or $\frac{15}{13}$.

§ 79. SOLUTION OF EQUATIONS OF THE FORM

$$mx^2+nx=b.$$

RULE.

Multiply the equation by 4 times the coefficient of x^2, and to the result add the square of the coefficient of x in the original equation. Then proceed as before.

EXAMPLES.

Ex. 1. Given $3x^2+7x=110$,
to find the values of x.

Multiplying by 12, we obtain

$$36x^2+84x=1320,$$

whence, by adding 49,

$$36x^2+84x+49=1369,$$

$\therefore$ $$6x+7=\pm 37,$$

and $$6x=-7\pm 37=30 \text{ or } -44,$$

consequently, $$x=5 \text{ or } -7\frac{1}{3}.$$

Ex. 2. Given $mx^2 + nx = b$,
to find the values of x.

First multiply by $4m$, and we have

$$4m^2x^2 + 4mnx = 4bm,$$

adding n^2,

$$4m^2x^2 + 4mnx + n^2 = 4bm + n^2.$$

Extracting the root,

$$2mx + n = \pm\sqrt{4bm + n^2},$$

and
$$x = \frac{-n \pm \sqrt{4bm + n^2}}{2m}.$$

Ex. 3. Given $3x^2 - 5x + 7 = 205$,
to find the values of x. *Ans.* $x = 9$ or $-7\frac{1}{3}$.

Ex. 4. Given $5x^2 + 12x = 47 + 19x - 23$,
to find the values of x. *Ans.* $x = 3$ or $-1\frac{3}{5}$.

Ex. 5. Given $4x - \frac{36 - x}{x} = 46$,
to find the values of x. *Ans.* $x = 12$ or $-\frac{3}{4}$.

Ex. 6. Given $\frac{10}{x} - \frac{3}{x+2} = \frac{10}{x+1}$,
to find the values of x. *Ans.* $x = 4$ or $-\frac{5}{3}$.

Ex. 7. Given $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}$,
to find the values of x. *Ans.* $x = 5$ or 2.

ON THE PROPERTIES OF THE ROOTS OF AN EQUATION.

§ **80.** If we examine the results in any of the preceding examples, we perceive that each equation of the second degree has two roots. The question may very naturally arise, whether these are the only ones of which such equations are susceptible.

§ **81.** If we transpose all the terms in any quadratic into the left-hand member, it will be of the form

$$mx^2 + nx + b = 0,$$

in which m, n, and b may be any numbers, positive or negative.

Thus the equation, Ex. 3, § 79, becomes, by this transposition,

$$3x^2 - 5x - 198 = 0.$$

The roots being $x = 9$, and $x = -7\frac{1}{3}$.

These last equations may be written

$$x - 9 = 0, \text{ and } x + 7\frac{1}{3} = 0 \text{ or } 3x + 22 = 0.$$

Now it will be found that

$$3x^2 - 5x - 198 = (x - 9)(3x + 22),$$

as may be proved by multiplication.

Since, then, $\quad 3x^2 - 5x - 198 = 0$

and $\quad (x - 9)(3x + 22) = 0$

are identical equations, it is evident that no values of x will satisfy one unless they do the other.

Moreover, since the product of two factors cannot be equal to 0, unless one of them is equal to zero, the latter of the two equations can only be satisfied, by making either

$$x - 9 = 0,$$

or

$$3x + 22 = 0,$$

which give the roots, 9 and $-7\frac{1}{3}$.

Hence it appears these are the only values of x.

§ **82.** The principles developed in the last article are of universal application.

They may be thus enunciated:

1st. *If a be a root of the equation*

$$mx^2 + nx + p = 0,$$

the expression, $mx^2 + nx + p$, is divisible by $x - a$.

2d. *Every quadratic equation has two roots, and but two.*

3d. If a and b be the roots of the equation

$$x^2 + mx + p = 0,$$

then will $\quad x^2 + mx + p = (x - a).(x - b).$

If now, we multiply the factors $x - a$ and $x - b$, we will have the product

$$x^2 - (a + b)x + ab,$$

comparing this with

$$x^2 + mx + p,$$

we find $\quad m = - (a + b)$ and $p = ab$.

Hence, *the second coefficient is the sum of the roots with their signs changed, and the absolute number is the product of the roots.*

§ **83.** If both roots are positive, that is, if

$$x = a, \text{ or } x = b.$$

the factors are

$$x - a, \text{ and } x - b.$$

Now $\quad (x - a)(x - b) = x^2 - (a + b)x + ab,$

in which the second coefficient is negative and the absolute term positive; that is, the equation is of the form

$$x^2 - mx + p = 0.$$

Equations of this form have therefore two positive roots.

Next let one of the roots, as b, be negative.

Then we have

$$x = a, \text{ and } x = - b,$$

or $\quad x - a = 0$, and $x + b = 0$.

Their product is

$$x^2 + (b - a)x - ab = 0,$$

If now $b > a$, this is of the form

$$x^2 + mx - p = 0.$$

Equations of this form have therefore one positive and one negative root, the negative being the greater.

If $a > b$, the equation

$$x^2 + (b - a)x - ab = 0,$$

is of the form $\quad x^2 - mx - p = 0.$

Equations of this class have therefore one positive and one negative root, the positive root being the greater.

Finally, let both the roots be negative.

That is, let $x = -a$, and $x = -b$,

or $x + a = 0$, and $x + b = 0$,

The product of these is

$$x^2 + (a + b)x + ab = 0,$$

which is of the form

$$x^2 + mx + p = 0.$$

Equations of this form have therefore two negative roots.

RECAPITULATION:

1st form.	$x^2 - mx + p = 0$,	two positive roots.
2d "	$x^2 + mx - p = 0$,	one positive and one negative, the negative being the greater.
3d "	$x^2 - mx - p = 0$,	one positive and one negative, the positive the greater.
4th "	$x^2 + mx + p = 0$,	two negative roots.

EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

§ **84.** When there are more unknown quantities than one, it is necessary first to eliminate all of these except one, and then solve the resulting equation by the principles already laid down. In some cases, however, the solution is facilitated by assuming a new unknown in place of some combination of those already employed, when, having determined the value of this, we may from that value readily determine those of the original unknowns.

EXAMPLES.

Ex. 1. Given $x + y = s$,

and $xy = p$,

to find the values of x and y.

Multiplying the first by x, we obtain

$$x^2 + xy = sx,$$

but $xy = p$,

$\therefore$ $x^2 + p = sx$,

whence $x^2 - sx = -p$,

and completing the square

$$4x^2 - 4sx + s^2 = s^2 - 4p.$$

Extracting the square root,

$$2x - s = \pm\sqrt{s^2 - 4p},$$

and $$x = \frac{s \pm \sqrt{s^2 - 4p}}{2},$$

consequently, $$y = s - x = \frac{s \mp \sqrt{s^2 - 4p}}{2}.$$

In this example, if $s^2 = 4p$, the values become

$$x = \frac{s}{2}, \text{ and } y = \frac{s}{2}.$$

But if s^2 is less than $4p$, the root

$$\sqrt{s^2 - 4p}$$

admits of no real value, and consequently, in that case, the question is impossible.

Ex. 2. Given $$x^2 + y^2 = 106,$$
and $$x - y = 4,$$
to find the values of x and y.

Here, from the second equation, we find

$$y = x - 4.$$

This value substituted in the first, gives

$$x^2 + x^2 - 8x + 16 = 106,$$

or $$2x^2 - 8x = 90,$$
dividing by 2, $$x^2 - 4x = 45,$$
whence $$x^2 - 4x + 4 = 49,$$
and $$x - 2 = \pm 7,$$
$\therefore$ $$x = 9 \text{ or } -5,$$
consequently, $$y = x - 4 = 4 \text{ or } -9.$$

Ex. 3. Given $$x^2 + y^2 + x + y = 86,$$
and $$xy - x - y = 23,$$
to find the values of x and y.

Adding twice the second to the first, we obtain

$$x^2 + 2xy + y^2 - x - y = 132. \qquad \text{(A)}$$

Let now $x+y=z,$
then $x^2+2xy+y^2=z^2.$

Substituting these values in (A), we obtain

$$z^2-z=132,$$

whence, completing the square,

$$4z^2-4z+1=529,$$

consequently, $2z-1=\pm 23,$
and $z=12$ or $-11.$

That is, $x+y=12$ or $-11.$

If this is added to the second equation, there results

$$xy=35 \text{ or } 12.$$

The equations $x+y=12,$
and $xy=35,$
solved as Ex. 1, give $x=7$ or $5,$
$y=5$ or $7.$

The others, $x+y=-11,$
and $xy=12,$
give in like manner, $x=\dfrac{-11\pm\sqrt{73}}{2},$

and $y=\dfrac{-11\mp\sqrt{73}}{2}.$

Ex. 4. Given $x^2-xy+y^2=109,$
and $xy+3x=81+3y,$
to find the values of x and y.

Transpose $3y$ in the second equation, and there results

$$xy+3x-3y=81,$$

whence, subtracting this from the first equation,

$$x^2-2xy+y^2-3x+3y=28,$$

put $x-y=z,$

Then $x^2-2xy+y^2=z^2,$
and $-3x+3y=-3z,$
consequently, $z^2-3z=28,$
and $4z^2-12z+9=121,$
$\therefore$ $2z-3=\pm 11,$
and $z=7$ or $-4.$

We have, therefore, $x - y = 7$ or -4,
and $xy = 81 - 3x + 3y = 60$ or 93.

From the equations $x - y = 7$,
and $xy = 60$,
there results, as in Ex. 1,

$$x = 12 \text{ or } -5,$$
$$y = 5 \text{ or } -12.$$

In like manner, the equations

$$x - y = -4,$$
and
$$xy = 93,$$
give
$$x = -2 \pm \sqrt{97}$$
and
$$y = +2 \pm \sqrt{97}$$

Ex. 5. Given
$$x + y + z = 7,$$
$$x^2 + y^2 + z^2 = 21,$$
and
$$xz = y^2,$$
to find the values of x, y, and z.

From the first we have, by transposing,

$$x + z = 7 - y,$$

whence squaring,

$$x^2 + 2xz + z^2 = 49 - 14y + y^2,$$
or because
$$xz = y^2,$$
$$x^2 + 2y^2 + z^2 = 49 - 14y + y^2,$$
but
$$x^2 + y^2 + z^2 = 21,$$
$\therefore$ by subtraction,
$$y^2 = 28 - 14y + y^2,$$
and
$$y = 2,$$
consequently, by substitution,

$$x + z = 5,$$
and
$$xz = 4,$$
whence, as in Ex. 1, we find,

$$x = 1,$$
$$z = 4,$$

consequently the results are

$$x = 1, y = 2, \text{ and } z = 4.$$

Ex. 6. Given $\frac{x^2}{y^2} + 4\frac{x}{y} = \frac{85}{9}$,

and $x - y = 2$,

to find the values of x and y.

Here, in the first equation, if we consider $\frac{x}{y}$ as the unknown, we shall have, by completing the square,

$$\frac{x^2}{y^2} + 4\frac{x}{y} + 4 = \frac{121}{9},$$

whence $\frac{x}{y} + 2 = \pm\frac{11}{3}$,

or $\frac{x}{y} = \frac{5}{3}$ or $-\frac{17}{3}$,

whence $x = \frac{5}{3}y$ or $-\frac{17}{3}y$.

These values substituted in the equation

$$x - y = 2,$$

give $\frac{5}{3}y - y = 2$,

or $-\frac{17}{3}y - y = 2$,

from which we obtain $y = 3$ or $-\frac{3}{10}$,

whence $x = 5$ or $\frac{17}{10}$.

Ex. 7. Given $x - y = 7$,

and $x^3 - y^3 = 1603$,

to find the values of x and y.

First Solution:

Assume $xy = p$,

then square the first equation, and there results

$$x^2 - 2xy + y^2 = 49.$$

Add $3xy = 3p$,

and $x^2 + xy + y^2 = 49 + 3p$.

Multiply this by the first equation, and we have

$$x^3 - y^3 = 343 + 21\,p,$$

$\therefore$ $$343 + 21\,p = 1603,$$

consequently, $$p = 60.$$

From the equations

$$x - y = 7,$$

and $$xy = 60,$$

we readily obtain $$x = 12 \text{ or } -5,$$

$$y = 5 \text{ or } -12.$$

Second Solution:

Divide the second equation by the first, the quotient is

$$x^2 + xy + y^2 = 229.$$

The square of the first is

$$x^2 - 2\,xy + y^2 = 49,$$

whence $$3\,xy = 180,$$

and $$xy = 60, \text{ as before.}$$

Third Solution:

Assume $$x = z + \frac{7}{2},$$

and $$y = z - \frac{7}{2},$$

then $$x^3 = z^3 + \frac{21}{2}z^2 + \frac{147}{4}z + \frac{343}{8},$$

and $$y^3 = z^3 - \frac{21}{2}z^2 + \frac{147}{4}z - \frac{343}{8},$$

$\therefore$ $$x^3 - y^3 = 21\,z^2 + \frac{343}{4} = 1603,$$

consequently $$84\,z^2 = 6069,$$

and $$4\,z^2 = 289,$$

whence $$2\,z = \pm 17,$$

and $$z = \pm \frac{17}{2},$$

consequently, $$x = z + \frac{7}{2} = \pm \frac{17}{2} + \frac{7}{2} = 12 \text{ or } -5,$$

and $$y = z - \frac{7}{2} = \pm \frac{17}{2} - \frac{7}{2} = 5 \text{ or } -12.$$

FOURTH SOLUTION:

From the first equation we obtain by transposition

$$x = 7 + y,$$

whence $$x^3 = 343 + 147\,y + 21\,y^2 + y^3,$$

$\therefore$ $$x^3 - y^3 = 343 + 147\,y + 21\,y^2 = 1603$$

Transposing and dividing by 21,

$$y^2 + 7\,y = 60,$$

$\therefore$ $$4\,y^2 + 28\,y + 49 = 289,$$

and $$2\,y + 7 = \pm 17,$$

whence $$y = 5 \text{ or } -12,$$

and $$x = 7 + y = 12 \text{ or } -5.$$

FIFTH SOLUTION:

Cube the first equation, there results

$$x^3 - 3\,x^2y + 3\,xy^2 - y^3 = 343,$$

but $$x^3 - y^3 = 1603,$$

$\therefore$ by subtraction, $$3\,x^2y - 3\,xy^2 = 1260,$$

divide by 3 times the first, we obtain

$$xy = 60,$$

from this and $$x - y = 7,$$

x and y are found as before.

Ex. 8. Given $$10\,x + y = 3\,xy,$$

and $$y - x = 2,$$

to find the values of x and y. *Ans.* $x = 2 \text{ or } -\dfrac{1}{3}$,

$y = 4 \text{ or } \dfrac{5}{3}$.

Ex. 9. Given $$x + y = 6,$$

and $$x^2y^2 + 4\,xy = 96,$$

to find the values of x and y. *Ans.* $x = 4 \text{ or } 2$,

$y = 2 \text{ or } 4$.

Consider xy as the unknown in the second equation, and complete the square.

Ex. 10. Given $x + 4y = 14,$
and $y^2 + 4x = 2y + 11,$
to find the values of x and y. *Ans.* $x = 2$ or $-46,$
$y = 3$ or $15.$

Ex. 11. Given $7x - 3y = 27,$
$3y^2 - 2x^2 = 3,$
to find the values of x and y. *Ans.* $x = 6$ or $\frac{120}{43},$
$y = 5$ or $-\frac{107}{43}.$

Ex. 12. Given $x^3 - y^3 = 98,$
and $xy = 15,$
to find the values of x and y. *Ans.* $x = 5$ or $-3,$
$y = 3$ or $-5.$

Ex. 13. Given $x^3 + y^3 = 407,$
and $x + y = 11,$
to find the values of x and y. *Ans.* $x = 7$ or $4,$
$y = 4$ or $7.$

This equation is solved precisely as Ex. 7.

Ex. 14. Given $\frac{x^2}{y} + \frac{y^2}{x} = 11\frac{2}{3},$
and $x + y = 10,$
to find the values of x and y. *Ans.* $x = 6, y = 4.$

Multiply the first by xy, and it becomes

$$x^3 + y^3 = \frac{35}{3}xy,$$

having also $x + y = 10.$

The example may be solved as Ex. 13.

Ex. 15. Given $x - y = 7,$
and $x^4 + y^4 = 21361,$
to find the values of x and y. *Ans.* $x = 12, y = 5.$

Solve this like Solution 3d, Ex. 7.

Ex. 16. Given $x - y = 3,$

$$\frac{x^2}{y} - \frac{y^2}{x} = 11\frac{7}{10},$$

to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

Ex. 17. Given $x^2 + y^2 = 34,$

and $x^2 - xy = 10,$

to find the values of x and y. *Ans* $x = 5$, $y = 3$.

§ 85. EXAMPLES PRODUCING QUADRATIC EQUATIONS.

Ex. 1. Required to find a number such that, when diminished by 10, and the remainder multiplied by the number itself, the product may be 39.

Here, if x represent the number, we shall have by the conditions,

$$(x - 10)x = 39,$$

that is, $x^2 - 10x = 39,$

whence $x^2 - 10x + 25 = 64,$

$$\therefore \quad x - 5 = \pm 8,$$

$$x = 13 \text{ or } -3.$$

Ex. 2. It is desired to enclose 10 acres of land in a rectangular form so that it shall only require 200 perches of fence to enclose it. What must be its length and breadth?

Let x be the length, then $100 - x$ will be the breadth; consequently,

$$x(100 - x) = 100x - x^2 = 1600 \text{ sq. perches.}$$

completing the square, $x^2 - 100x + 2500 = 900,$

whence, $x - 50 = \pm 30,$

and $x = 80 \text{ or } 20,$

$$100 - x = 20 \text{ or } 80.$$

Ex. 3. Divide 12 into two parts, such that the square of the one may be 4 times the square of the other.

Ans. 8 and 4.

Ex. 4. A man bought some cloth for \$280, there being thirty-three more yards than there were dollars in the cost per yard. How many yards did he purchase?

Ans. 40 yds.

Ex. 5. Divide a line, 30 inches in length, into two parts, such that the rectangle of the whole and one part may equal the square of the other part. *Ans.* $\begin{cases} 18.54 + \text{ins.} \\ 11.45 + \text{ins.} \end{cases}$

Ex. 6. A butcher bought as many calves as cost him $60; he reserved 5 and sold the remainder for $45, having gained 50 cents a head by them. How many did he buy? *Ans.* 15.

Ex. 7. Two merchants purchased together 550 yds. of cloth for $1500, each having paid as many cents per yard as there were yards bought by the other. How much did each buy? *Ans.* 250 yds. and 300 yds.

Ex. 8. A man being asked his age, replied, When I was born, my mother was 20 years old, and her present age, multiplied by mine exceeds the sum of our ages by 2500. What was his age? *Ans.* 42 yrs.

Ex. 9. What two numbers are those whose sum is 15, and the sum of their squares 117? *Ans.* 9 and 6.

Ex. 10. There are two square fields, containing together 318.8 acres, the difference of their sides being 30 chains. What are their dimensions?

Ans. 52 chains and 22 chains.

Ex. 11. Required to determine a number from the following considerations, viz.: The number itself, divided by the sum of its digits, is equal to the tens digit, and if 36 be added to the number, the digits will be inverted.

Ans. 48.

NOTE.—If x represent the tens and y the units, the value of the number is $10x + y$. Similarly, if x be the hundreds, y the tens, and z the units digit, the number will be $100x + 10y + z$.

Ex. 12. Required two numbers whose difference, added to the difference of their squares, makes 150, and whose sum, added to the sum of their squares, makes 330.

Ans. 15 or -16, and 9 or -10.

Ex. 13. There is a number consisting of three digits, such that the sum of the squares of the digits is 104; but the square of the tens digit exceeds twice the product of the others by 4, and if 594 be added to the number, its digits will be reversed. What is the number? *Ans.* 268.

Ex. 14. Two lights, whose intensities are in the ratio of 9 to 25, illuminate an object placed in a line joining them. How far from the brighter must it be placed, so as to receive an equal degree of illumination from them both, they being 50 inches apart, and the intensity of light being inversely as the square of the distance? *Ans.* $31\frac{1}{4}$ inches.

Ex. 15. The sum of two numbers is 16, and the sum of their cubes 1072. What are those numbers?

Ans. 7 and 9.

Ex. 16. The sum of two numbers is 7, and the sum of their fourth powers 641. Required the numbers.

Ans. 2 and 5.

Ex. 17. The product of two numbers is 120, and if the greater be diminished by 5, and the less increased by 7, their product will be 150. What are they?

Ans. 15 and 8 or $-5\frac{5}{7}$, and -21.

Ex. 18. What two numbers are those whose sum is 15, and the sum of their fifth powers 103125?

Ans. 10 and 5.

Ex. 19. A. and B. bought 41 oxen, each paying $420. Now, A. having bargained for the better lot, which are worth $1 a head more than the others, required the number each must take? *Ans.* A. 20 and B. 21.

Ex. 20. Divide the number 5 into two such parts, that each being divided by the other, the sum of the quotients is $2\frac{1}{6}$. *Ans.* The parts are 3 and 2.

Ex. 21. A gentleman has a rectangular court-yard, 100 feet long and 80 feet broad, and wishes to put a gravel walk half round it, so that it shall occupy one-sixth of the ground. Required the breadth of the walk. *Ans.* 7.741 feet.

CHAPTER VII.

PROPORTION AND PROGRESSION.

PROPORTION.

§ **86.** RATIO is the relation which one quantity bears to another in magnitude. This relation is expressed by the quotient arising from the division of the first of the two given quantites by the second. Thus the ratio of 9 to 3 is $9 \div 3 = 3$; of 3 to 9 is $3 \div 9 = \frac{1}{3}$; of $a : b$ is $a \div b$.

To indicate that two quantities are compared in this manner, they are written with a colon between them, thus: $3 : 5$, $a : b$; which are read 3 to 5 and a to b, express the ratio which 3 bears to 5 and a bears to b.

§ **87.** The first term of a ratio is called the *antecedent*, and the second the *consequent*.

§ **88.** The ratio which the product of the antecedents of several ratios bears to the product of the consequents is said to be *compounded* of those ratios.

Thus $ace : bdf$, is said to be compounded of $a : b$, $c : d$, and $e : f$.

§ **89.** A ratio compounded of two equal ratios, is called the *duplicate* ratio of either of them. A ratio compounded of three equal ratios is called a *triplicate* ratio, &c.

§ **90.** When four quantities are so related that the first has to the second the same ratio as the third has to the fourth, they are *proportionals*. The series of terms constitues a *proportion*.

Thus 4, 12, 3, and 9 are proportionals, the ratio of the first to the second, and of the third to the fourth being each $\frac{1}{3}$.

The equality of two ratios is indicated by placing a double colon between them. Thus the above proportion would be written,

$$4 : 12 : : 3 : 9$$

and read, As 4 is to 12, so 3 is to 9.

In some old treatises, the sign of equality is used instead of the double colon. Thus, $4 : 12 = 3 : 9$.

Cor. From the above definition it is clear that if four quantities are proportionals, they will be proportionals by *inversion*, that is, the first consequent will have to its antecedent the same ratio as the second consequent has to its antecedent.

§ **91.** Quantities so related that every term has the same ratio to the succeeding term, are in *continued proportion*. Thus 2, 6, 18, 54, 162, &c., are in continued proportion.

A series of this kind is likewise called a *geometrical progression*, or *progression by quotient*.

§ **92.** From Section 90, it is plain that the quotient of the first term of a proportion divided by the second, equals that of the third term divided by the fourth.

Thus if $m : n : : o : p$ we shall have $\frac{m}{n} = \frac{o}{p}$, since these fractions express the ratios of m to n and o to p; conversely, if $\frac{m}{n} = \frac{o}{p}$ we shall have $m : n : : o : p$.

§ **93.** The value of a ratio is not altered if both the terms be multiplied or both be divided by the same number. Thus the ratios $2 : 6$, $4 : 12$, $6 : 18$, are all equal. So likewise are $a : b$ and $ma : mb$.

§ **94.** If four numbers are proportionals, the product of the extremes is equal to that of the means; and conversely, if the product of the extremes of four numbers equals that of the means, they are proportionals.

Let $$a : b : : c : d,$$
then $ad = bc$, for we have (§ 92)
$$\frac{a}{b} = \frac{c}{d},$$
clearing of fractions, $ad = bc$.

Again, let $mn = rs$, then $m : r : : s : n$, for dividing by rn, we have
$$\frac{m}{r} = \frac{s}{n}$$
or $$m : r : : s : n \text{ (§ 92)}.$$

§ **95.** If three quantities are in continued proportion, the product of the extremes equals the square of the mean. For let a, b, and c be in continued proportion, then

$$a : b :: b : c \text{ (§91) and } ac = b^2 \text{ (§94).}$$

Hence the geometric mean of two numbers equals the square root of their product.

§ **96.** If the corresponding terms of any number of proportions be multiplied together the products will be proportional.

Thus, let $a : b :: c : d$
$e : f :: g : h$
$i : k :: l : m$

Then will $aei : bfk :: cgl : dhm$.

For (§92) we have

$$\frac{a}{b} = \frac{c}{d}, \frac{e}{f} = \frac{g}{h}, \text{ and } \frac{i}{k} = \frac{l}{m}$$

$$\therefore \quad \frac{aei}{bfk} = \frac{cgl}{dhm}$$

or (§92) $aei : bfk :: cgl : dhm$.

The above property may be otherwise enunciated; thus, Ratios compounded of equal ratios are equal.

Cor. From the above it is apparent that if four quantities are proportionals, and equimultiples of the antecedents and likewise of the consequents be taken, the results will be proportionals.

Thus, if $a : b :: c : d$, $ma : nb :: mc : nd$.

§ **97.** If three quantities be in continual proportion, the first has to the third the *duplicate* ratio of that which it has to the second.

Thus, if $a : b :: b : c$, $a : c$ is the duplicate ratio of $a : b$.

For (§88) the ratio compounded of the ratios $a : b$ and $b : c$ is $ab : bc$, which (§93) equals $a : c$. But since $a : b :: b : c$, the ratio compounded of these ratios is the duplicate ratio of $a : b$ (§89).

Cor. Similarly, the ratio of the first of four continued proportionals to the fourth is the *triplicate* ratio of the first to the second.

§ **98.** The first of three continued proportionals has to the third the same ratio which the square of the first has to the square of the second.

Let $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For since $a : b :: b : c$

and $a : b :: a : b$ we have (§96)

$$a^2 : b^2 :: ab : bc :: a : c \text{ (§93)}$$

Cor. Similarly, if a, b, c, d, e, &c., be a series of continued proportionals we shall have

$$a : c :: a^2 : b^2,\ a : d :: a^3 : b^3,\ a : e :: a^4 : b^4$$

and generally if p be the nth term $a : p :: a^{n-1} : b^{n-1}$.

From the above it is plain that the duplicate ratio of two numbers is the ratio of their squares.

§ **99.** If A, B, C, D, &c., and a, b, c, d, &c., be two series of quantities such that $A : B :: a : b$, $B : C :: b : c$, &c., then will the first of the first series have to the last of the first series the same ratio as the first of the second series has to the last.

For since
$$A : B :: a : b$$
$$B : C :: b : c$$
$$C : D :: c : d$$

we shall have (§96) $ABC : BCD :: abc : bcd$

or (§93) $A : D :: a : d$

§ **100.** If the consequents of one proportion be the same as those of another, then will the sum or difference of the first antecedents have to the first consequent the same ratio as the sum or difference of the second antecedents has to the second consequent.

Thus, if $a : b :: c : d$

and $e : b :: f : d$

we shall have $a \pm e : b :: c \pm f : d$.

For since $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{b} = \frac{f}{d}$ (§92)

we shall have $\frac{a \pm e}{b} = \frac{c \pm f}{d}$,

or (§92) $a \pm e : b :: c \pm f : d$.

Cor. Similarly, if the antecedents of a series of proportionals be the same, the first antecedent will have to the sum of the first consequents the same ratio as the second antecedent has to the sum of the second consequents.

§101. When four quantities are proportionals, they are proportionals by composition; that is, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

Let $a : b :: c : d$, then will $a + b : b :: c + d : d$.

For since $a : b :: c : d$

and $b : b :: d : d$

we have (§100) $a + b : b :: c + d : d$.

§102. If four quantities are in proportion they are in proportion by division; that is, the difference between the first and second is to the second as the difference between the third and fourth is to the fourth.

Let $a : b :: c : d$, then $a - b : b :: c - d : d$

For since $a : b :: c : d$ and $b : b :: d : d$,

we have (§100) $a - b : b :: c - d : d$.

§103. If $a : b :: c : d$,

we shall have $a + b : a - b :: c + d : c - d$

For, (§101) $a + b : b :: c + d : d$, (A)

and (§102) $a - b : b :: c - d : d$

whence (§90 *cor.*) $b : a - b :: d : c - d$, from (A)

and this we have (§99) $a + b : a - b :: c + d : c - d$.

§104. If four like quantities be proportionals, they will be proportionals by alternation; that is, the first will have to the third the same ratio which the second has to the fourth.

Thus, let $a : b :: c : d$ then $a : c :: b : d$,

For since $a : b :: c : d$

and $b : c :: b : c$

we have (§96) $ab : bc :: bc : cd$,

whence $a : c :: b : d$.

§105. If any number of like magnitudes be proportionals, one antecedent will be to its consequent as the sum of the antecedents is to the sum of the consequents.

Let $a : b :: c : d :: e : f$, then $a : b :: a + c + e : b + d + f$.

For since $a : b :: c : d$, and $a : b :: e : f$,

we have (§104) $a : c :: b : d$

and $a : e :: b : f$,

but $a : a :: b : b$,

$\therefore$ (§100 *cor.*) $a : a + e + c :: b : b + d + f$,

and alternately (§104) $a : b :: a + c + e : b + d + f$.

§106. If four quantities be proportionals, their like powers, and also their like roots, will be proportional.

Let $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$.

For since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a^n}{b^n} = \frac{c^n}{d^n}$,

and $a^n : b^n :: c^n : d^n$.

ARITHMETICAL PROGRESSION.

§107. An *arithmetical progression*, or progression by difference, is a series, the successive terms of which increase or decrease by a common difference. Thus the series,

2, 5, 8, 11, 14, 17, &c.

and 57, 54, 51, 48, 45, 42, &c.

are in *arithmetical progression;* the common difference in each case being 3.

The common difference of a descending series is generally regarded as negative.

§108. If a be the first term of a series, and d the common difference, the series will be

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \&c.$$

d being negative for a decreasing series.

By examining the terms, we perceive that the coefficient of d in any term is one less than the number of the term.

Thus in the fourth term we have $3d$, &c. Hence, if l be the nth term, we shall have

$$l = a + (n - 1)d.$$

Therefore, to find any term of the series,

Subtract one from the number of the term, multiply the remainder by the common difference, and to the product add the first term.

EXAMPLES.

Ex. 1. Find the 20th term of the series

3, 8, 13, &c.?

Here the common difference is 5;

$$\therefore \quad l = 3 + 19 \times 5 = 3 + 95 = 98.$$

Ex. 2. What is the 15th term of the series

11, 17, 23, &c.? *Ans.* 95.

Ex. 3. Find the 12th term of the series

50, 47, &c. *Ans.* 17.

Ex. 4. The first term being $a - 5b$, and the common difference b; what is the 17th term? *Ans.* $a + 11b$.

Ex. 5. What is the 35th term of the series

3, $\frac{10}{3}$, $\frac{11}{3}$, &c.? *Ans.* $14\frac{1}{3}$.

Ex. 6. Find the 150th term of the series

47, $45\frac{1}{2}$, &c. *Ans.* $-176\frac{1}{2}$.

§**109.** *The arithmetical mean between two numbers is equal to half their sum.*

Let a, b, and c represent three numbers in arithmetical progression, then:

$$b - a = c - b,$$

when by transposition $\quad 2b = a + c,$

and $\quad b = \frac{1}{2}(a + c).$

§ **110.** *If four quantities be in arithmetical progression, the sum of the extremes is equal to that of the means.*

Let a, b, c, and d represent four such quantities, then:

$$b - a = d - c,$$

when by transposition $b + c = a + d$.

§ **111.** *The sum of the extremes is equal to the sum of any two terms equally distant from them.*

Let $\qquad a,\ a + d,\ a + 2\,d$, &c.

be the series commencing with the least. If l be the last term, then will

$$l,\ l - d,\ l - 2\,d, \text{ \&c.}$$

be the same series beginning with the greatest. Hence as the sum of any pair equally distant from the two extremes is $a + l$, the truth of the proposition is evident.

Since $l = a + (n - 1)\,d$, $a + l = 2\,a + (n - 1)\,d$.

§ **112.** To find the sum of the series, let

$$S = a + (a + d) + (a + 2\,d) + (a + 3\,d) \text{ \&c.}$$

n being the number of terms.

Writing the series backwards we have, l being the last term,

$$S = l + (l - d) + (l - 2\,d) + (l - 3\,d) \text{ \&c.}$$

whence, by addition,

$$2\,S = (a + l) + (a + l) + (a + l) + \text{\&c.} = n\,(a + l).$$

Consequently, $\qquad S = \frac{n}{2}\,(a + l);$ $\qquad$ (A)

but $\qquad a + l = 2\,a + (n - 1)\,d,$ $\qquad$ (§111).

therefore, $\qquad S = \frac{n}{2}\,(2\,a + (n - 1)\,d).$ $\qquad$ (B)

From (A) and (B) we derive the following

RULE FOR SUMMING AN ARITHMETICAL PROGRESSION.

1st. *Multiply the sum of the extremes by half the number of terms;* or,

2d. *Multiply the common difference by the number of terms less one, and add the product to twice the first term.*

Multiply the sum thus obtained by half the number of terms; the product will be the sum of the series.

EXAMPLES.

Ex. 1. The first term being 3, the common difference 5, and the number of terms 20, what is the sum?

Here, $l = 3 + 19 \times 5 = 3 + 95 = 98,$

and $S = \frac{n}{2}(a + l) = 10\,(3 + 98) = 10 \times 101 = 1010,$

or, without finding l,

$$S = \frac{n}{2}(2a + (n-1)\,d) = 10\,(6 + 19 \times 5) =$$

$$10 \times 101 = 1010.$$

Ex. 2. Required the sum of the first 25 terms of the series 99, 97, 95, &c.

Here, the common difference is -2.

Here $S = \frac{n}{2}(2a + (n-1)\,d) = \frac{25}{2}(198 - 24 \times 2)$

$$= \frac{25}{2} \times 150 = 1875.$$

Ex. 3. What is the sum of the first 15 terms of the series 1, 3, 5, &c. *Ans.* 225.

Ex. 4. What is the sum of n terms of the same series? *Ans.* n^2.

Ex. 5. Required the sum of the first 50 terms of the series 2, 4, 6, &c. *Ans.* 2550.

Ex. 6. What is the sum of n terms of the same series? *Ans.* $n^2 + n$.

Ex. 7. Required the sum of 17 terms of the series 70, $69\frac{2}{3}$, &c. *Ans.* $1144\frac{2}{3}$.

Ex. 8. Required the sum of n terms of the series 1, 5, 9, &c. *Ans.* $2n^2 - n$.

Ex. 9. How far will a man travel in 10 days, if he goes 10 miles the first day, 13 the second, and so on, increasing 3 miles every day? *Ans.* 235 *m.*

Ex. 10. A gentleman sold a horse on the following conditions, viz. he was to have \$1 for the first nail in his shoes, \$5 for the second, and so on. Now there being 8 nails in each shoe, what was the price of the horse? *Ans.* \$2016.

§113. To find the common difference when the first and last terms and the number of terms are given, we have the following formula, viz.

$$d = \frac{l - a}{n - 1} \qquad (\S 108). \qquad (A)$$

If the first term, the sum of the series, and the number of terms are given, we have (B, §112)

$$d = \frac{\frac{2S}{n} - 2a}{n - 1} = \frac{2(S - na)}{n^2 - n} \qquad (B)$$

To find the number of terms, we have (§112)

$$S = \frac{n}{2}(2a + (n - 1)d) = an + \frac{n^2 d}{2} - \frac{nd}{2}$$

whence, by transposition,

$$dn^2 + (2a - d)n = 2S. \qquad (C)$$

This equation, being solved by quadratics, will give n.

Examples.

Ex. 1. The first term of an arithmetical progression is 3; the common difference 4; and the sum 300. Required the number of terms.

Here, $d = 4$, $a = 3$, and $S = 300$.

Formula (C) therefore becomes

$$4n^2 + 2n = 600,$$

whence $n = 12.$

Ex. 2. Given the sum of the series 3240, the number of terms 40, and the first term 3; to find the common difference.

Ans. 4.

Ex. 3. What is the common difference, the first term being 5, the last term 45, and the number of terms 21?

Ans. 2.

Ex. 4. Required the number of terms, the first term being 11, the common difference 3, and the sum of the series 595.

Ans. 17.

GEOMETRICAL PROGRESSION.

§ **114.** A series of numbers is said to be in geometrical progression (§ 91) when every term has the same ratio to the succeeding term.

Thus, 3, 6, 12, 24, &c.

is a series in geometrical progression, the ratio of any two successive terms being $\frac{1}{2}$.

§ **115.** The *Inverse Ratio* ratio is generally considered the *ratio* in geometrical progression. Thus in the above series the *ratio* is 2.

§ **116.** If $a =$ the first term and $r =$ the ratio, then the series is evidently

$$a, ar, ar^2, ar^3, \&c.;$$

the index of the power of r in any term being less by unity than the number of the term. Thus the fourth term is ar^3, the ninth ar^8, consequently the nth term $l = ar^{n-1}$.

If the series were diminishing, the ratio would be a proper fraction.

Hence, to find the last term, we have the following

RULE.

Raise the ratio to a power whose index is less by unity than the number of terms, and multiply the result by the first term.

EXAMPLES.

Ex. 1. The first term being 4, the ratio 2, required the 10th term.

Here $l = ar^{n-1} = 4 \times 2^9 = 4 \times 512 = 2048$.

Ex. 2. Required the 20th term of the series

2, 6, 18, &c.

Ans. 2324522934.

Ex. 3. Required the 12th term of the series

7, 14, 28, &c.

Ans. 14336.

Ex. 4. $125 being placed at compound interest at 6 per cent. for seven years, what is the amount?

In this example, the amount of $1 for one year being $1.06, the amount of $125 will be $125 \times (1.06)$. The amount for the second year will be $125 \times (1.06)^2$.

Hence the several amounts will be the successive terms of a geometric series, the first term being the principal, the ratio the amount of $1 for 1 year, and the number of terms being a number greater by unity than the number of years.

In this example,

$$\text{Amount} = 125 \times (1.06)^7 = 125 \times 1.503630 = 187.954.$$

The rule in this case may be expressed thus:

Call the amount of 1 *dollar for* 1 *year the ratio. Raise the ratio to a power whose index is the number of years, and multiply the principal by the result.*

Ex. 5. What is the amount of $100 for 20 years at 6 per cent. compound interest?

Ans. $320.713.

Ex. 6. Required the amount of $325 for 11 years at 5 per cent. compound interest.

Ans. $555.860.

§**117.** To find the sum of a geometric series, we may proceed thus:

Let $\quad S = ar^{n-1} + ar^{n-2} \ldots\ldots ar + a.$

Multiply both members of the equation by $r - 1$, and we have

$$S\ (r - 1) = ar^n - a = a\ (r^n - 1),$$

$$\therefore \qquad S = \frac{ar^n - a}{r - 1} = a\,\frac{r^n - 1}{r - 1} \qquad \text{(A)}$$

If r is a proper fraction, the numerators and denominators of these fractions will be negative. We may therefore write them thus:

$$S = \frac{a - ar^n}{1 - r} = a\ \frac{1 - r^n}{1 - r} \qquad \text{(B)}$$

And since $\qquad ar^n = lr,$

l being the last term, we have

$$S = \frac{rl - a}{r - 1} \text{ or } \frac{a - rl}{1 - r}$$

according as the series is increasing or decreasing.

Rule.

Raise the ratio to a power whose index is the number of terms. Divide the difference between this power and unity by the difference between the ratio and unity, and multiply the quotient by the first term, the product will be the sum of the series.

Note.—In a decreasing series continued to infinity, the last term is 0. Hence to sum such a series, divide the first term by the difference between the ratio and unity.

Examples.

Ex. 1. Required the sum of the first 10 terms of the series 3, 6, 12, &c.

Here $r = 2$. Hence $S = a\,\dfrac{r^n - 1}{r - 1} = 3\ (1024 - 1) = 3069.$

Ex. 2. Required the sum of 15 terms of the series

$$4,\ 12,\ 36,\ \&c.$$

Here $S = a\dfrac{r^n - 1}{r - 1} = 4.\dfrac{3^{15} - 1}{3 - 1} = 4.\dfrac{14348906}{2}$

$$= 28697812.$$

Ex. 3. Required the sum of the first 19 terms of the series

$$\tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{18} - \tfrac{1}{54} + \&c.$$

Here $r = -\dfrac{1}{3}$ $\therefore$ $r^{19} = -\dfrac{1}{1162261467}$

$$\therefore \quad S = a\frac{1 - r^n}{1 - r} = \frac{1}{2}\cdot\frac{1 + \dfrac{1}{1162261467}}{1 + \frac{1}{3}}$$

$$= \frac{1}{2}\cdot\frac{1162261468}{1162261467} \div \frac{4}{3} = \frac{290565367}{774840978}$$

Ex. 4. Required the sum of the series

$7 + 21 + 63$, &c. to 8 terms. *Ans.* 22960.

Ex. 5. What is the sum of the series

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, to 14 terms? *Ans.* $1\frac{8191}{8192}$.

Ex. 6. Sum the first 9 terms of the series

$6 - 18 + 54$, &c. *Ans.* 29526.

Ex. 7. What is the sum of the first 10 terms of the same series? *Ans.* $-$ 88572.

Ex. 8. Required the sum of the series

$$2,\ \tfrac{2}{3},\ \tfrac{2}{9},\ \text{to infinity.}$$

Here $$S = \frac{a}{1 - r} = \frac{2}{\frac{2}{3}} = 3$$

Ex. 9. What is the sum of

$3+\frac{3}{10}+\frac{3}{100}$ ad infinitum? *Ans.* $3\frac{1}{3}$.

Ex. 10. Sum the series

$3+2+\frac{4}{3}$ to infinity. *Ans.* 9.

Ex. 11. What is the sum of the series

$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$ to infinity? *Ans.* $\frac{2}{3}$.

§118. To find the ratio, we have the following formula, viz.

$$r=\sqrt[n-1]{\frac{l}{a}}$$

which is easily obtained from the following,

$$l=ar^{n-1}. \quad (\S 116).$$

Examples.

Ex. 1. Insert 5 geometric means between 3 and 192.

Here there will be 7 terms.

Hence $\qquad r=\sqrt[6]{\frac{192}{3}}=\sqrt[6]{64}=2,$

and the means are 6, 12, 24, 48, and 96.

Ex. 2. Insert 4 geometric means between 7 and 1701.

Ans. 21, 63, 189, and 567.

Ex. 3. Insert 4 geometric means between $\frac{1}{2}$ and $-\frac{16}{3125}$.

Ans. $-\frac{1}{5}$, $\frac{2}{25}$, $-\frac{4}{125}$, and $\frac{8}{625}$.

Ex. 4. A man travelled five miles the first hour, and having increased his hourly rate in geometrical progression, finds that during the fifth hour he has gone $15\frac{65}{81}$ miles. What was the rate of increase, and how far would he travel during the 7th hour? *Ans.* Rate $1\frac{1}{3}$, distance 7th hour $28\frac{68}{729}$.

§119. By the principles of geometrical progression we may readily determine the amount of an annuity at compound interest.

For it is evident that if P be the annuity and r the inte-

rest of 1 dollar for 1 year, the amounts of the several payments, beginning with the last, will be (Ex. 4, § 116.)

$$P, \ P(1+r) \ P(1+r)^2 \ P(1+r)^3 \ldots . P(1+r)^{n-1};$$

and since these amounts form a geometrical progression, it will only be necessary to obtain the sum by the rule, § 117.

The formula may be written thus:

$$S = P\frac{(1+r)^n - 1}{(1+r) - 1} = P\frac{(1+r)^n - 1}{r}$$

Rule.

Call the amount of 1 dollar for 1 year the ratio. Raise the ratio to a power whose index equals the number of years; and from this power subtract 1. Divide the remainder by the interest of 1 dollar for a year, and multiply the quotient by the annuity.

Examples.

Ex. 1. Required the amount of an annuity of $300 which has been foreborne 5 years; interest being reckoned at 5 per cent.

Here the ratio is 1.05.

$$\therefore \quad S = 300 \times \frac{(1.05)^5 - 1}{.05} = 300 \times \frac{1.2762815625 - 1}{.05}$$

$$= 300 \times 5.52563125 = 1657.689375 \text{ dollars.}$$

Ex. 2. What is the amount of an annuity of $400 for 8 years, at 6 per cent. *Ans.* $3958.988.

Ex. 3. What is the amount of an annuity of $225 for 9 years at 4 per cent. *Ans* $2381.128.

§ 120. Examples involving Arithmetical and Geometrical Progression.

Ex. 1. There are three numbers in arithmetical progression, whose sum is 21, and the sum of the first and second is to the sum of the second and third as 3 is to 4.

Here, let $x - y \quad x$, and $x + y$ represent the numbers, and we shall have

$$x - y + x + x + y = 21,$$

or $$3x = 21,$$

whence $$x = 7,$$

likewise, $$x - y + x : x + x + y :: 3 : 4,$$

or $$2x - y : 2x + y :: 3 : 4,$$

whence $$8x - 4y = 6x + 3y,$$

$\therefore$ $$7y = 2x = 14,$$

and $$y = 2.$$

The numbers are therefore

5, 7, and 9.

Ex. 2. The sum of three numbers in arithmetical progression is 324, and the first is to the third as 5 to 7. What are the numbers? *Ans.* 90, 108, and 126.

Ex. 3. The distance between two towns is 340 miles. Now, if a train of cars starts from one and travels uniformly at the rate of 20 miles per hour towards the other, how long will it be before it meets another, which left the second at the same time, and goes 5 miles the first hour, 7 miles the second, and so on in arithmetical progression?

Ans. 10 hours.

Ex. 4. The product of the extremes of four numbers in arithmetical progression is 34, and that of the means 84 What are the numbers? *Ans.* 2, 7, 12, and 17.

Ex. 5. The product of the extremes of four numbers in arithmetical progression is 45, and the sum of the means 18. What are the numbers? *Ans.* 3, 7, 11, and 15.

Ex. 6. The sum of the first and third of four numbers in geometrical progression is 78, and that of the second and fourth 390. What are the numbers?

Ans. 3, 15, 75, and 375.

Ex. 7. The sum of the extremes of four numbers in Geometrical progression is 130, and the sum of the means 40 What are the numbers? *Ans.* 2, 8, 32, and 128.

Ex. 8. The sum of three numbers in geometrical progression is 7, and the sum of their squares 21. What are the numbers? *Ans.* 1, 2, and 4.

Ex. 9. There are three numbers in geometrical progression, whose sum is 14, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers. *Ans.* 2, 4, 8.

Ex. 10. A number consists of three digits in arithmetical progression; being divided by the sum of its digits gives 48, and if 198 be subtracted from it, the digits will be inverted. What is the number? *Ans.* 432.

Ex. 11. The difference of the extremes of four numbers in geometrical progression is 248, and the difference of the means 40. What are they? *Ans.* 2, 10, 50, and 250.

Ex. 12. The sum of the extremes of four numbers in geometrical progression is 65, and the product of the means 64. What are the numbers? *Ans.* 1, 4, 16, and 64.

Ex. 13. $700 was divided among four persons, their shares being in geometrical progression, and the difference between the greatest and least being to the difference between the means as 37 to 12. Required the shares.

Ans. 108, 144, 192, and 256.

Ex. 14. The first term of an arithmetical progression is 5, the common difference 3, and the sum 1455. Required the number of terms. *Ans.* 30.

Ex. 15. The sum of an arithmetical progression is 2910, the first term 10, and the number of terms 30. What is the common difference? *Ans.* 6.

Ex. 16. The sum of a geometrical progression is 242, the number of terms 5, and the ratio 3. What is the first term? *Ans.* 2.

Ex. 17. The difference of the extremes of four numbers in geometrical progression is 52, and the product of the means 108. Required the numbers.

Ans. 2, 6, 18, and 54

THE END.